MAGIC063: Differentiable Manifolds
Lecture 1

Gerasim Kokarev
The University of Leeds

6 October, 2020
Comments of the reading list


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  - topological space, Hausdorff space, base of topology;
  - metric spaces, their topology, bases;
  - quotient topology;
  - topological manifolds.
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- These notions from point-set topology will be recalled.
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- These notions from point-set topology will be recalled.
- Q&A pause and a possible poll.
Definition. A topological space is a pair $(X, \tau)$, where $X$ is a set and $\tau$ is a collection of its subsets that satisfies the following properties:

1. $\emptyset \in \tau$, $X \in \tau$;
2. If $U_\alpha \in \tau$, where $\alpha \in A$, then $\bigcup_\alpha U_\alpha \in \tau$;
3. If $U, V \in \tau$, then $U \cap V \in \tau$.

A set $U \in \tau$ is called open. A subset $V \subset X$ is called closed, if the complement $X \setminus V$ is open.

Trivial examples of topology on every set $X$:
- trivial topology, given by $\tau = \{\emptyset, X\}$;
- discrete topology, given by $\tau = 2^X$, set of all subsets.

Definition. A topological space $(X, \tau)$ is called Hausdorff if every two points can be separated in the following sense: for any $x, y \in X$, $x \neq y$, there exists open subsets $U, V \subset X$ such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. 
Overview of the point-set topology

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Definition. A topological space \((X, \tau)\) is called *compact* if any covering \(\{U_\alpha\}\) of \(X\) by open subsets (that is \(\bigcup U_\alpha = X\)) contains a finite subcovering \(U_{\alpha_1}, \ldots, U_{\alpha_N}\) (that is \(\bigcup_{i=1}^{N} U_{\alpha_i} = X\)).
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- **Exercise 1.** Let \(u : (X, \tau_X) \to (Z, \tau_Z)\) be a continuous surjective map. Show that if \(X\) is compact, then so is \(Z\).

- **Exercise 2.** Let \((X, \tau_X)\) be a compact topological space, and \((Z, \tau_Z)\) be a Hausdorff space. Show that any continuous bijective map \(u : X \to Z\) is a homeomorphism.
Definition. Let \((X, \tau)\) be a topological space. A subset \(\beta \subset \tau\) is called the base of the topology \(\tau\) if for any \(U \in \tau\) there exists \(\{U_\alpha\}\), \(U_\alpha \in \beta\) such that \(U = \bigcup_\alpha U_\alpha\).
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Proposition 1. Let \(\beta\) be a collection of subsets of \(X\) such that:

- \(\emptyset \in \beta\);
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Then \(\beta\) is a basis of the topology \(\tau(\beta)\) on \(X\) whose open sets are unions of sets in \(\beta\).
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Then $\beta$ is a basis of the topology $\tau(\beta)$ on $X$ whose open sets are unions of sets in $\beta$. 

Example 1. (Topology of a real line.) The standard topology on $\mathbb{R}$ can be defined by choosing the base $\beta_1 = \{(a, b) : a < b, a, b \in \mathbb{R}\}$, or the base $\beta_2 = \{(a, b) : a < b, a, b \in \mathbb{Q}\}$. 

Proof. One needs to check that $\tau(\beta)$ satisfies the axioms of topology. Left as an exercise.
Definition. Let \((X, \tau)\) be a topological space. A subset \(\beta \subset \tau\) is called the base of the topology \(\tau\) if for any \(U \in \tau\) there exists \(\{U_\alpha\}\), \(U_\alpha \in \beta\) such that \(U = \bigcup U_\alpha\).

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Proof. One needs to check that \(\tau(\beta)\) satisfies the axioms of topology. Left as an exercise.
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**Proposition 2.** Let $(X, d)$ be a metric space. Then the collection of open balls $\beta = \{B(x, r) : x \in X, r > 0\}$ (together with $\emptyset$) is a base of a topology on $X$. 

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Conclusion: any metric space is automatically a topological space; besides, the defined topology is always Hausdorff.
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  - \(d(x, y) = d(y, x)\) for all \(x, y \in X\);
  - \(d(x, y) \leq d(x, z) + d(y, z)\) for all \(x, y, z \in X\).

- By \(B(x, r)\) we below mean the open ball \(B(x, r) = \{y \in X : d(x, y) < r\}\).

- Proposition 2. Let \((X, d)\) be a metric space. Then the collection of open balls \(\beta = \{B(x, r) : x \in X, r > 0\}\) (together with \(\emptyset\)) is a base of a topology on \(X\).

- **Proof.** One needs to check that \(\beta\) satisfies the hypotheses of Proposition 1.

- **Conclusion:** any metric space is automatically a topological space; besides, the defined topology is always Hausdorff.

- **Trivial examples:**
  - The trivial topology is not Hausdorff, and hence, is not metrizable;
  - The discrete topology is induced by the metric

\[
d(x, y) = \begin{cases} 
0, & \text{if } x = y; \\
1, & \text{if } x \neq y.
\end{cases}
\]
Final questions
THE END