**Definition**

A (topological) manifold of dimension \( m \) is a topological space \( M \) such that each point \( P \in M \) admits a neighborhood \( U \) homeomorphic to an open subset \( V \subset \mathbb{R}^m \).

\( U \), together with a homeomorphism \( \phi : U \rightarrow V \subset \mathbb{R}^m \), is called a **chart**. A collection of charts covering \( M \) is called an **atlas**.

Fixing a chart \( U \) is equivalent to defining local coordinates \((x^1, \ldots, x^m)\) on \( U \):

\[
\phi(P) = (x^1, \ldots, x^m) \in V \subset \mathbb{R}^m
\]

Thus, basically we may say that \( M \) is a manifold if in a neighborhood of any point \( P \) we can introduce local coordinates. Of course, this correspondence

\[
P \leftrightarrow (x^1, \ldots, x^m) = \text{coordinates of } P
\]

must be a bijection continuous in both directions (i.e., a homeomorphism). We shall always assume two additional conditions:

- \( M \) is a Hausdorff space, i.e., any two points \( P, Q \in M, P \neq Q \), have disjoint neighborhoods.
- \( M \) admits a countable atlas
Terminology: *Smooth* = of class $C^\infty$

Motivation: Having local coordinates $(x^1, \ldots, x^m)$ we may work with our manifold (at least locally, i.e., in $U$) in the same way as we do it in $\mathbb{R}^m$. However, the following “problem” appears: some points $P$ belong to several charts $U_1, U_2, \ldots$. In other words, we have several choices for local coordinates. Clearly, we want them to be equivalent and we want to have the possibility to change them freely. In differential geometry, one of the main principles is that ”properties of the objects we work with should not depend on the choice of local coordinates”. For example, if we need to verify the smoothness of a certain object (function, vector field, etc.), we would like to be able to do this in *any local coordinate system* we wish.

This idea can be conceptualised in the following definition:

**Definition**
A manifold $M$ is called *smooth* if the transition functions between any two local coordinate systems are smooth.
More precisely: Let $U$ and $U'$ be two intersecting charts, then on their intersection $U \cap U'$ we have two different coordinate systems $(x^1, \ldots, x^m)$ and

$(x'^1, \ldots, x'^m)$ and we can naturally define the transition functions between them

$$x^1' = h_1(x^1, \ldots, x^m), \ldots, x^m' = h_m(x^1, \ldots, x^m)$$

Formally, these functions can be defined as (the components of) the map

$$\phi' \circ \phi^{-1} : \phi(U \cap U') \rightarrow \phi'(U \cap U').$$

The smoothness of $M$ means that the functions

$$h_1(x^1, \ldots, x^m), \ldots, h_m(x^1, \ldots, x^m)$$

are smooth in usual sense (and it is so for any two intersecting charts).
\[ \phi(P) = (x^1, \ldots, x^m) \]

Transition functions

\[ \phi' = \phi \circ \phi^{-1} \]
Examples of manifolds

- Vector space $\mathbb{R}^n$
- Any open subset in $\mathbb{R}^n$
- Graph of a smooth function (map)
- Two-dimensional surfaces
- Spheres $S^n$ and Projective spaces $\mathbb{R}P^n$
- Lie Groups
- Homogeneous spaces of Lie groups
- Subsets in $\mathbb{R}^n$ given by a system of equations (with certain “regularity” condition)
In \( \mathbb{R}^n \), consider a system of equations:

\[
\begin{align*}
&f_1(x_1, \ldots, x_n) = 0, \\
&\vdots \\
&f_k(x_1, \ldots, x_n) = 0.
\end{align*}
\]

where \( f_i \) are smooth and \( k \leq n \). Let \( M \subset \mathbb{R}^n \) be the set of solutions.

**Regularity condition:**

The rank of the Jacobi matrix

\[
J = \begin{pmatrix}
\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial f_k}{\partial x_1} & \frac{\partial f_k}{\partial x_2} & \cdots & \frac{\partial f_k}{\partial x_n}
\end{pmatrix}
\]

is maximal (that is, \( = k \)) at any point \( P \in M \).

**Theorem**

*If the regularity condition holds, then \( M \) carries the natural structure of a smooth manifold of dimension \( n - k \).*
**Example**

Consider the group $O(3)$ as a subset in $\mathbb{R}^9$ (the space of $3 \times 3$-matrices). The condition $A \cdot A^\top = Id$ is a matrix equation which is equivalent to the system of 6 usual equations with 9 unknowns:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \quad \quad \rightarrow \quad \begin{cases} f_1 : & a_{11}^2 + a_{12}^2 + a_{13}^2 = 1 \\ f_2 : & a_{11}a_{21} + a_{12}a_{22} + a_{13}a_{23} = 0 \\ f_3 : & a_{11}a_{31} + a_{12}a_{32} + a_{13}a_{33} = 0 \\ f_4 : & a_{21}^2 + a_{22}^2 + a_{23}^2 = 1 \\ f_5 : & a_{21}a_{31} + a_{22}a_{32} + a_{23}a_{33} = 0 \\ f_6 : & a_{31}^2 + a_{32}^2 + a_{33}^2 = 1 \end{cases}$$

**Jacobi matrix $J$:**

\[ df_1 = \begin{pmatrix} 2a_{11} & 2a_{12} & 2a_{13} & * & * & * & * & * & * \\ a_{21} & a_{22} & a_{23} & * & * & * & * & * & * \\ a_{31} & a_{32} & a_{33} & * & * & * & * & * & * \\ 0 & 0 & 0 & 2a_{21} & 2a_{22} & 2a_{23} & * & * & * \\ 0 & 0 & 0 & a_{31} & a_{32} & a_{33} & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 2a_{31} & 2a_{32} & 2a_{33} \end{pmatrix} \]
Since the rows of $A$ are linearly independent (recall that $\det A = \pm 1$), the same is true for the Jacobi matrix $J$. So the rank of $J$ is maximal and equal to 6. According to the implicit function theorem, the equation $AA^\top = Id$ define a smooth manifold of dimension $9 - 6 = 3$ in $\mathbb{R}^9$.

Example
Consider the surface $V$, a cone, in $\mathbb{R}^3$ given by $x^2 + y^2 - z^2 = 0$. Compute the differential:

$$df = (2x, 2y, -2z).$$

The differential is nowhere zero except the point $O = (0, 0, 0)$. The implicit function theorem guarantees that locally $V$ has a structure of a smooth manifold at any $P \in V$ different from $O$. It is easy to see that $O = (0, 0, 0)$ is indeed singular in the sense that there is no neighborhood of $O$ in $V$ homeomorphic to a 2-disc.

Conclusion: $V$, as a whole, is not a smooth manifold because of the singular point $O$, this is exactly the point where the regularity condition of the implicit function theorem fails.
cone
\[ xe^2 + y^2 - z^2 = 0 \]
good point
regularity condition \( df \neq 0 \) holds

Singular point
regularity condition fails

\( O = (0, 0, 0) \)
Maps between manifolds

Definition
Let $F : M \to N$ be a (continuous) map between two smooth manifolds. This map is called *smooth (or $C^\infty$)*, if it is so in local coordinates. More precisely, this means the following. Let $P \in M$ and $Q = F(P) \in N$ be its image. Consider local coordinates $(x^1, \ldots, x^m)$ and $(y^1, \ldots, y^n)$ in neighborhoods of $P$ and $Q$ respectively. Then locally $F$ can be written in these local coordinates as:

$$(y^1, \ldots, y^n) = F(x^1, \ldots, x^m), \text{ i.e., } F = \begin{cases} 
  y^1 = f_1(x^1, \ldots, x^m) \\
  y^2 = f_2(x^1, \ldots, x^m) \\
  \vdots \\
  y^n = f_n(x^1, \ldots, x^m)
\end{cases}$$

The smoothness of $F$ means that all the functions $f_1, \ldots, f_n$ are smooth.

Remark
Since the transition functions between charts are smooth, the above definition does not depend of the choice of local coordinates $(x^1, \ldots, x^m)$ and $(y^1, \ldots, y^n)$ in neighborhoods of $P$ and $Q$. We may verify the smoothness condition in any local coordinates we wish.

Definition
A *diffeomorphism* $F : M \to N$ is a smooth bijective map such that its inverse $F^{-1} : N \to M$ is also smooth.
Tangent vectors

The are several different ways to introduce the notion of a tangent vector. We shall use the following simple idea. If we deal with a manifold $M$ embedded in $\mathbb{R}^n$, then a tangent vector at a point $P \in M$ can be defined as just a tangent vector to a certain smooth curve $\gamma(t) \subset M$ passing through $P$.
The set of all possible tangent vectors at $P$ is then the tangent space to $M$ at $P$.

Let $\gamma(t)$ be a smooth curve in $M$, i.e., a smooth map $\gamma : (-\epsilon, \epsilon) \to M$. In local coordinates, this map is given as $\gamma(t) = (x^1(t), \ldots, x^m(t))$. Then the tangent vector to $\gamma$ at point $P = \gamma(0)$ is defined to be simply

$$\frac{d\gamma}{dt}(0) = \left( \frac{dx^1}{dt}(0), \frac{dx^2}{dt}(0), \ldots, \frac{dx^m}{dt}(0) \right)$$

Thus, in local coordinates, any tangent vector $\xi$ is given as an $m$-tuple $\xi = (\xi^1, \ldots, \xi^m)$, here $m = \dim M$.
The problem, however, is that this definition depends on the choice of local coordinates. What happens to $(\xi^1, \ldots, \xi^m)$ if we change local coordinates $(x^1, \ldots, x^m) \to (x^1', \ldots, x^m')$?
Standard computation: the tangent vector to $\gamma(t)$ in the new coordinates is

$$\left( \frac{dx^1}{dt}, \ldots, \frac{dx^m}{dt} \right) = \left( \sum_{i=1}^m \frac{\partial x^1}{\partial x^i} \frac{dx^i}{dt}, \ldots, \sum_{i=1}^m \frac{\partial x^m}{\partial x^i} \frac{dx^i}{dt} \right)$$

In other words, the “new” coordinates of the same tangent vector $\xi$ are:

$$(\xi^1, \ldots, \xi^m) = \left( \sum_{i=1}^m \frac{\partial x^1}{\partial x^i} \xi^i, \ldots, \sum_{i=1}^m \frac{\partial x^m}{\partial x^i} \xi^i \right) \quad (1)$$

**Definition**

A **tangent vector** $\xi$ at a point $P$ is defined in any local coordinate system $(x^1, \ldots, x^m)$ as an $m$-tuple $(\xi^1, \ldots, \xi^m)$; in addition, it is required that the transformation law for the components of $\xi$ under coordinate change $(x^1, \ldots, x^m) \rightarrow (x'^1, \ldots, x'^m)$ is given by (1).

**Remark**

Notice that the transformation $(\xi^1, \ldots, \xi^m) \rightarrow (\xi^1, \ldots, \xi^m)$ (at a fixed point $P$) is linear, whereas the transformation $(x^1, \ldots, x^m) \rightarrow (x'^1, \ldots, x'^m)$ is not.
Definition
A linear mapping $A : C^\infty(M) \to \mathbb{R}$ is called a derivation (first order differential operator) at point $P \in M$ if it satisfies the Leibnitz rule:

$$A(f \cdot h) = A(f) \cdot h(P) + A(h) \cdot f(P).$$

The relationship between tangent vectors and derivations is very simple and natural: Each tangent vector $\varepsilon$ defines a derivation of this kind (called directional derivative along $\varepsilon$) by:

$$A_{\varepsilon}(f) = \left. \frac{d}{dt} \right|_{t=0} f(\gamma(t)) = \sum_i \frac{\partial f}{\partial x^i} \varepsilon^i,$$

where $\gamma(t)$ is a smooth curve such that $\gamma(0) = P$, $\frac{d\gamma}{dt}(0) = \varepsilon$.

It is easy to see that the correspondence $\varepsilon \mapsto A_{\varepsilon}$ is a natural bijection between tangent vectors and derivations. Hence:

Definition
A tangent vector to $M$ at point $P$ is defined to be a derivation at $P$. Usually, we shall denote $A_{\varepsilon}$ simply by $\varepsilon$ (since we identify them): $\varepsilon(f) = \sum_i \varepsilon^i \frac{\partial f}{\partial x^i}$.

Notation: $\varepsilon = \sum_i \varepsilon^i \frac{\partial}{\partial x^i}$. 