General linear group $GL(n, \mathbb{R})$

$GL(n, \mathbb{R}) = \{ X \in n \times n \text{ matrix} : \det X \neq 0 \}$

Properties of $GL(n, \mathbb{R})$:

- $\dim GL(n, \mathbb{R}) = n^2$;
- $GL(n, \mathbb{R})$ is not compact;
- $GL(n, \mathbb{R})$ is not connected but consists of two connected components
  $GL_+(n, \mathbb{R}) = \{ X : \det X > 0 \}$ and $GL_-(n, \mathbb{R}) = \{ X : \det X < 0 \}$;
- the Lie algebra $gl(n, \mathbb{R})$ consists of all $n \times n$ matrices.
Special linear group $SL(n, \mathbb{R})$

$SL(n, \mathbb{R}) = \{ X \in n \times n \text{ matrix} : \det X = 1 \}$

Properties of $SL(n, \mathbb{R})$:

- $\dim SL(n, \mathbb{R}) = n^2 - 1$;
- $SL(n, \mathbb{R})$ is not compact;
- $SL(n, \mathbb{R})$ is connected;
- the Lie algebra $sl(n, \mathbb{R})$ consists of $n \times n$ matrices with zero trace; $sl(n, \mathbb{R}) = \{ A : \text{tr} A = 0 \}$;
- $SL(n, \mathbb{R})$ is a normal subgroup in $GL(n, \mathbb{R})$ and the quotient group $GL(n, \mathbb{R})/SL(n, \mathbb{R})$ is isomorphic to $\mathbb{R}^*$ (the multiplicative subgroup in $\mathbb{R}$, i.e., $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ under multiplication).
- $SL(n, \mathbb{R})$ is not simply connected, $\pi_1(SL(n, \mathbb{R})) = \mathbb{Z}_2$ for $n > 2$, and $\pi_1(SL(2, \mathbb{R})) = \mathbb{Z}$. 
The Lie group associated with a bilinear form

Let \( B \) be a bilinear form on \( \mathbb{R}^n \):

\[
B : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}, \quad u, v \mapsto B(u, v), \quad \text{where } u, v \in \mathbb{R}^n, B(u, v) \in \mathbb{R}.
\]

Standard examples: inner product, symplectic form.
Each bilinear form \( B \) can naturally be given by its matrix \( B = (b_{ij}) \):

\[
B(u, v) = \sum_{i,j} b_{ij} u_i v_j = (u_1 \ldots u_n) \begin{pmatrix} b_{11} & \ldots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{n1} & \ldots & b_{nn} \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}
\]

Consider the set of all invertible linear transformations \( X : \mathbb{R}^n \rightarrow \mathbb{R}^n \) that preserve the form \( B \).

\[
G_B = \{ X \in GL(n, \mathbb{R}) : B(Xu, Xv) = B(u, v) \text{ for all } u, v \in \mathbb{R}^n \}
\]

Here we identify non-degenerate matrices \( X \in GL(n, \mathbb{R}) \) with invertible linear transformations \( X : \mathbb{R}^n \rightarrow \mathbb{R}^n \).
The fact that \( G_B \) is a group is standard: the set of invertible transformations preserving "something" is always a group.
Proposition

$G_B$ is an algebraic linear group (and, therefore, a Lie subgroup in $GL(n, \mathbb{R})$).

In matrix notation, $G_B$ is defined by one matrix equation (quadratic in $X$):

$$G_B = \{ X \in GL(n, \mathbb{R}) : X^\top BX = B \}.$$  

The corresponding Lie algebra $\mathfrak{g}_B = T_E G_B$ is

$$\mathfrak{g}_B = \{ A \in gl(n, \mathbb{R}) : A^\top B + BA = 0 \}.$$  

Proof. The equation $X^\top BX = B$ is just a reformulation of $\mathcal{B}(Xu, Xv) = \mathcal{B}(u, v)$ in terms of matrices. Thus, $G_B$ is algebraic. To describe its Lie algebra $\mathfrak{g}_B$ we need to verify the following property:

$$A^\top B + BA = 0 \quad \text{if and only if} \quad (\exp(tA))^\top B \exp(tA) = B$$

$\iff$ follows just from differentiating $(\exp(tA))^\top B \exp(tA) = B$ at $t = 0$.  

$\implies$ can be obtained from the following argument:

$A^\top B + BA = 0$ implies $BA = (A^\top)^\top B$, hence by induction $BA^n = (A^\top)^n B$ and therefore:

$$B \exp(tA) = B \sum_n \frac{t^n A^n}{n!} = \sum_n \frac{t^n (-A^\top)^n}{n!} B = \exp(-tA^\top) B$$

and finally $(\exp(tA))^\top B \exp(tA) = \exp(tA^\top) \exp(-tA^\top) B = B$, as required.
In general, there is no restriction on $B$: this form may be symmetric or skew-symmetric, neither symmetric nor skew-symmetric, degenerate or non-degenerate.

The properties of $G_B$ essentially depend on the properties of $B$, as we shall see below. However, the following statement holds:

**Proposition**

If bilinear forms $B_1$ and $B_2$ are equivalent in the sense that $B_1 = C^\top B_2 C$ for a certain invertible matrix $C$ (this means that these two forms are related by a suitable change of coordinates), then the corresponding groups $G_{B_1}$ and $G_{B_2}$ are isomorphic and the isomorphism is given by conjugation:

$$\Phi : G_{B_1} \rightarrow G_{B_2}, \quad X \mapsto \Phi(X) = CXC^{-1}.$$  

**Proof.** The conjugation $X \mapsto CXC^{-1}$ is an isomorphism between any Lie group $G$ and its image $\Phi(G)$. Thus, we only need to prove that $\Phi(G_{B_1}) = G_{B_2}$. Let $X \in G_{B_1}$, then

$$(CXC^{-1})^\top B_2 CXC^{-1} = (C^{-1})^\top X^\top C^\top B_2 CXC^{-1} =

(C^{-1})^\top X^\top B_1 XC^{-1} = (C^{-1})^\top B_1 C^{-1} = B_2.$$  

Thus, $CXC^{-1} \in G_{B_2}$, i.e. $\Phi(G_{B_1}) \subset G_{B_2}$. The proof that $G_{B_2} \subset \Phi(G_{B_2})$ is similar.
Orthogonal groups $O(n)$ and $SO(n)$

$O(n)$ is a particular case of $G_B$, namely, $B = E$ (identity matrix).

$$O(n) = \{X \in GL(n, \mathbb{R}) : X^\top X = E\}$$

Properties of $O(n)$:

- $\dim O(n) = \frac{n(n-1)}{2}$;
- $O(n)$ is compact;
- $O(n)$ consists of two connected components; the connected component of the identity is $SO(n) = \{X \in O(n) : \det X = 1\}$;
- $SO(n)$ is not simply connected, $\pi_1(SO(n)) = \mathbb{Z}_2$ for $n > 2$, and $\pi_1(SO(2)) = \mathbb{Z}$;
- the Lie algebra $so(n)$ consists of skew-symmetric matrices;
- each one-parameter subgroup in $SO(n)$ is conjugate to

$$X(t) = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_k \end{pmatrix} \quad \text{where } X_i \text{ is either } \begin{pmatrix} \cos \varphi_i t & -\sin \varphi_i t \\ \sin \varphi_i t & \cos \varphi_i t \end{pmatrix} \text{ or } 1$$
Pseudo-orthogonal groups $O(p, q)$ and $SO(p, q)$

$O(p, q)$ is a particular case of $G_B$ for

$$B(u, v) = u_1 v_1 + \cdots + u_p v_p - u_{p+1} v_{p+1} - \cdots - u_{p+q} v_{p+q},$$

or, equivalently,

$$B = E_{p, q} = \text{diag}(1, \ldots, 1, -1, \ldots, -1)$$

We assume that $p + q = n$ so that $B$ is non-degenerate. In addition, neither $p$ nor $q$ is 0 (if $p$ or $q$ is zero, then $O(p, q) = O(n)$).

**Important case:** $O(1, 3)$ known as the **Lorentz group**.

**Properties of $O(p, q)$:**

- $\dim O(p, q) = \frac{n(n-1)}{2}$, $n = p + q$;
- $O(p, q)$ is not compact;
- $O(p, q)$ is not connected and consists of 4 connected components;
- $SO(p, q) = O(p, q) \cap SL(n, \mathbb{R})$ consists of 2 components;
- $\det X = \pm 1$ for $X \in O(p, q)$;
- the Lie algebra $so(p, q)$ consists of the matrices $A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$ where $A_1, A_2, A_3, A_4$ are submatrices of dimension $p \times p$, $p \times q$, $q \times p$ and $q \times q$ respectively, such that $A_1$ and $A_4$ are skew-symmetric and $A_3^\top = A_2$.

Equivalently, this simply means that $E_{p, q} A$ is skew-symmetric.
Symplectic group $Sp(2n, \mathbb{R})$

$Sp(2n, \mathbb{R})$ is a group $G_J$ of linear transformations that preserve a bilinear non-degenerate skew-symmetric form $J$. Usually, as the matrix of $J$ one takes:

$$J = J_{2n} = \begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix}$$

so that

$$Sp(2n, \mathbb{R}) = \{ X \in GL(2n, \mathbb{R}) \mid X^\top JX = J \}$$

Notice that in this situation, our vector space has to be even-dimensional, i.e., $\mathbb{R}^{2n}$, since odd-dimensional skew-symmetric matrices are necessarily degenerate.

Properties of $Sp(2n, \mathbb{R})$:

- $\dim Sp(2n, \mathbb{R}) = n(2n + 1)$;
- $Sp(2n, \mathbb{R})$ is not compact;
- $Sp(2n, \mathbb{R})$ is connected;
- $\det X = 1$ for $X \in Sp(2n, \mathbb{R})$;
- the Lie algebra $sp(2n, \mathbb{R})$ consists of the matrices $A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}$ where $A_1, A_2, A_3, A_4$ are submatrices of dimension $n \times n$ such that: $A_1 = -A_4^\top$ and $A_3$ and $A_2$ are symmetric. Equivalently: $JA$ is symmetric.
- $Sp(2n, \mathbb{R})$ is not simply connected and $\pi_1(Sp(2n, \mathbb{R})) = \mathbb{Z}$. 