1. Consider the Lie group $G_B = \{ X \in GL(3, \mathbb{R}) \mid X^\top B X = B \}$, where $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

In other words, $G_B$ is the group of invertible transformations that preserve the (degenerate) symmetric form given by $B$.

- What is $\dim G_B$?
- Describe all matrices $X \in G_B$.
- Describe the corresponding Lie algebra $\mathfrak{g}_B$.
- How many connected components are there in $G_B$?
- Prove that the identity component $(G_B)_0$ is diffeomorphic to $S^1 \times \mathbb{R}^3$.

2. Consider the matrix group $C(L) = \{ X \in GL(n, \mathbb{R}) \mid X^{-1} LX = L \}$, where $L$ is a certain $n \times n$ matrix (viewed as an operator but not a bilinear form); $C(L)$ is usually called the centralizer of $L$.

Prove that $\dim C(L) \geq n$ for any $L$.

Describe $C(L)$ for $L = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ assuming that all $\lambda_i$’s are distinct. What happens if some of $\lambda_i$’s coincide?

Describe $C(L)$ in the case when $L$ is a Jordan block of maximal dimension:

$$ L = \begin{pmatrix} \lambda & 1 & \ldots & 0 \\ 0 & \lambda & 1 & 0 \\ \vdots & \vdots & \ddots & 1 \\ 0 & 0 & \ldots & \lambda \end{pmatrix} $$

3. Consider the space $V$ of skew symmetric $n \times n$ matrices (over $\mathbb{R}$) endowed with the bilinear operation

$$ [A_1, A_2]_C = A_1 CA_2 - A_2 CA_1, \quad A_1, A_2 \in V, $$

where $C$ is a fixed symmetric matrix. Prove that this operation determines the structure of a Lie algebra on $V$. Prove that if $C$ is positive definite, then $(V, [\cdot, \cdot]_C)$ is isomorphic to $so(n)$.

What happens if in this construction we interchange “symmetric” and “skew symmetric” (i.e., for $C$ skew symmetric and $A_1, A_2$ symmetric)?

4. Prove that the symplectic group $Sp(2n, \mathbb{R})$ is connected. Show that $Sp(2, \mathbb{R}) = SL(2, \mathbb{R})$.

5. Let $B$ be a positive definite symmetric bilinear form on $\mathbb{R}^{2n}$ and $J$ be a non-degenerate skew symmetric form on the same $\mathbb{R}^{2n}$. Denote their matrices (w.r.t. a
certain basis in $\mathbb{R}^n$, the same for both of them) by $B$ and $J$ respectively, and consider the Lie groups that “preserve” $B$ and $J$ (see Lecture 8):

$$G_B = \{X \in GL(n, \mathbb{R}) \mid X^\top BX = B\} \text{ and } G_J = \{X \in GL(n, \mathbb{R}) \mid X^\top JX = J\}.$$ 

We know that $G_B \simeq O(n)$ and $G_J \simeq Sp(n, \mathbb{R})$ (here $\simeq$ means “isomorphic as Lie groups”).

The question is to analyze the intersection $G_B \cap G_J$ depending on $B$ and $J$.

- Prove: $G_{B+J} = G_B \cap G_J$.
- Prove that for standard $B$ and $J$ (i.e., $B = E_{2n}$, $J = \begin{pmatrix} 0 & E_n \\ -E_n & 0 \end{pmatrix}$), the intersection $G_B \cap G_J = O(2n) \cap Sp(2n, \mathbb{R})$ is isomorphic to the unitary group $U(n)$.
- Describe the intersection $G_B \cap G_J$ for $B = E_{2n}$ and $J = \begin{pmatrix} 0 & D_n \\ -D_n & 0 \end{pmatrix}$ where $D = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$ with distinct $\lambda_i$.
- Describe (up to isomorphism) $G_B \cap G_J$ for $B$ and $J$ arbitrary. (Use the following result: for any skew-symmetric matrix $J$ there exist an orthogonal matrix $P$ such that $P^{-1}JP = P^\top JP = \begin{pmatrix} 0 & D_n \\ -D_n & 0 \end{pmatrix}$ as above but some of $\lambda_i$’s may coincide.)
- What is the minimal possible dimension for $G_C$, where $C$ is an arbitrary bilinear form on $\mathbb{R}^{2n}$?