# FUNCTIONAL ANALYSIS
# via MAGIC

# LINEAR ALGEBRA (Topic 1)

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This section aims to cover a decent proportion of what might be termed 'Linear Algebra for Functional Analysts'. The vector spaces of interest to us are primarilty infinite-dimensional, and for this it is useful to develop some further language beyond that met in the finite-dimensional context. In particular, infinite dimensionality encourages departure from a reliance on numbers as indices to a use of more general sets that are not necessarily totally ordered.
Features of our overview include, constant reference to the paradigm examples of spaces of scalar-valued (and vector-valued) functions and their subspace of functions having finite (cardinality) support on the one hand, and an emphasis on universal properties on the other hand. A central theme is that of 'linearisation'. We shall indulge a fairly detailed treatment of linear tensor products, since this topic is often absent from undergraduate courses and, after all, if you can add (vector spaces) surely you will want to multiply them too!
Topics not covered here, due to time and space constraints, include the theory of Fredholm maps (invertible modulo the ideal of finite rank maps).
There is an appendix on nonunital algebras, since these arise naturally in analysis. For example the algebra of compact operators on a Banach space is nonunital (unless the space is finite dimensional), as is the algebra of continuous complex-valued functions which vanish at infinity, defined on a locally compact (but non-compact) Hausdorff topological space such as the real line.

*Notations.* Throughout, $\mathbb{K}$ denotes one of the two fields $\mathbb{C}$ or $\mathbb{R}$. The complex conjugate of $z \in \mathbb{C}$ is denoted $z^*$. For a set $S$, its *power set* $\mathcal{P}(S)$ is the set of all subsets of $S$. For sets $S$ and $T$, $\mathcal{F}(S; T)$ denotes the set of all functions with source/domain $S$ and target/codomain $T$. A brief glossary of further notations is given at the end.

## LA-1

- Vector operations on subsets of a vector space
- Quotient vector spaces
- Canonical factorisation of a linear map

# Vector operations on subsets of a vector space

Let $V$ be a vector space over $\mathbb{K}$. For $A, B \subset V$, $\Lambda \subset \mathbb{K}$, $x \in V$ and $\mu \in \mathbb{K}$,

$$\begin{aligned}
&A + B := \{u + v : u \in A,\ v \in B\},\\
&A - B := \{u - v : u \in A,\ v \in B\},\\
&\Lambda A := \{\lambda u : \lambda \in \Lambda,\ u \in A\},\\
&A + x := \{u + x : u \in A\},\\
&x + A := \{x + u : u \in A\} \quad \text{and}\\
&\mu A := \{\mu u : u \in A\}.
\end{aligned}$$

Thus $A + B = B + A$ and $A + x = A + \{x\}$; these define maps

$$\mathcal{P}(V) \times \mathcal{P}(V) \to \mathcal{P}(V) \quad \text{and} \quad \mathcal{P}(\mathbb{K}) \times \mathcal{P}(V) \to \mathcal{P}(V),$$

respectively $\mathcal{P}(V) \times V \to \mathcal{P}(V)$ etc.

### Definition (Convex hull and Linear span)

Let $A$ be a subset of a vector space $V$ over $\mathbb{K}$. The *convex hull* of $A$ is given by

$$\operatorname{Conv} A := \bigcup_{n\in\mathbb{N},\, t\in\mathbb{R}_+^n,\, \sum t_i \;=1} \Big(t_1 A + \cdots + t_n A\Big);$$

its *linear span* is given by

$$\operatorname{Lin} A := \{0\} \cup \mathbb{K}A \cup \big(\mathbb{K}A + \mathbb{K}A\big) \cup \big(\mathbb{K}A + \mathbb{K}A + \mathbb{K}A\big) \cup \cdots$$

### Trivial example

For $u, v \in V$, the *line segment*

$$[u, v] := \Big\{(1-t)u + tv : t \in [0, 1]\Big\}$$

equals $\operatorname{Conv}\{u, v\}$.

### Proposition (Linear quotient spaces)

*Let $U$ be a subspace of a vector space $V$ over $\mathbb{K}$. For the equivalence relation on $V$ given by*

$$v_1 \sim v_2 \quad \text{if} \quad (v_2 - v_1) \in U$$

*the prescriptions*

$$[v] + [v'] := [v + v'] \quad \text{and} \quad \lambda[v] := [\lambda v] \quad (v, v' \in V,\ \lambda \in \mathbb{K})$$

*give a well-defined vector space structure to the quotient set (of equivalence classes) with respect to which the quotient map is linear.*

### Proof.

Straightforward verification (exercise). □

### Remarks

- The resulting vector space over $\mathbb{K}$ is denoted $V/U$.
- Equivalence classes are of the form $U + v$ ($v \in V$) and the vector space operations in $V/U$ correspond precisely to addition and scalar multiplication of subsets of $V$—with the single exception of scalar multiplication by $0_{\mathbb{K}}$.

# Canonical factorisation of linear maps

**Theorem (First Isomorphism Theorem of Linear Algebra)**

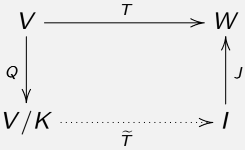
*Let $T \in L(V; W)$ for vector spaces $V$ and $W$ over $\mathbb{K}$ and set*

$$K := \operatorname{Ker} T \quad \textit{and} \quad I := \operatorname{Ran} T.$$

*Then there is a unique map $\widetilde{T} \in L(V/K; I)$ such that*

$$T = J \circ \widetilde{T} \circ Q$$

*where $Q$ is the quotient map $V \to V/K$ and $J$ is the inclusion map $I \to W$; moreover $\widetilde{T}$ is an isomorphism.*



## **LA-2** (for self-study)

- Conjugate-linear maps
- Linear involutions
- Conjugate vector spaces

For vector spaces $U$ and $V$ over $\mathbb{C}$, we set

$$L_{\mathrm{conj}}(U;V) := \{J : U \to V \mid J \text{ is conjugate-linear}\}.$$

### Remarks

- $L_{\mathrm{conj}}(U;V)$ is a vector space over $\mathbb{C}$, just as $L(U;V)$ is, and equally its elements have a canonical factorisation.
- $L(V;W)L_{\mathrm{conj}}(U;V) \subset L_{\mathrm{conj}}(U;W)$; $L_{\mathrm{conj}}(V;W)L(U;V) \subset L_{\mathrm{conj}}(U;W)$.
- $L_{\mathrm{conj}}(V;W)L_{\mathrm{conj}}(U;V) \subset L(U;W)$.
- If $J \in L_{\mathrm{conj}}(U;V)$ is bijective then $J^{-1} \in L_{\mathrm{conj}}(V;U)$.

### Definition (Linear involution)

Let $V$ be a vector space over $\mathbb{C}$. Then $J \in L_{\mathrm{conj}}(V)$ is called a *linear involution* on $V$ if it enjoys the involutive property $J^2 = I_V$.

### Example

Term-wise complex conjugation gives a linear involution on $M_{nm}(\mathbb{C})$.

### Definition

Let $V$ be a vector space over $\mathbb{C}$. A (*vector space*) *conjugation* of $V$ is a complex vector space $W$ with a conjugate-linear isomorphism $J : V \to W$.

### Proposition (Existence and uniqueness of conjugate spaces)

*Every complex vector space $V$ has a conjugation. Moreover, if $(W_1, J_1)$ and $(W_2, J_2)$ are two conjugations of $V$ then there is a unique linear isomorphism $J_{21} : W_1 \to W_2$ satisfying $J_{21}J_1 = J_2$.*

### Proof.

Uniqueness is trivial: set $J_{21} = J_2 J_1^{-1}$. For existence set

$$W = \{\widetilde{v} : v \in V\} \subset L_{\mathrm{conj}}(V^{\mathrm{dual}}; \mathbb{C}) \text{ and } J : v \mapsto \widetilde{v} \text{ where } \widetilde{v}(\varphi) := \varphi(v)^*.$$

That $J$ is injective, and thus a (conjugate-linear) isomorphism, follows from the fact (proved below) that $\varphi(v) = 0$ for all $\varphi \in V^{\mathrm{dual}}$ implies $v = 0$ ($V^{\mathrm{dual}}$ separates the elements of $V$). □

We therefore speak of "the" conjugate space of a complex vector space.

**Example (Conjugate space of $L(U; V)$)**

Let $(U^*, u \mapsto u^*)$ and $(V^*, v \mapsto v^*)$ be conjugate spaces of complex vector spaces $U$ and $V$. Then

$$(L(U^*; V^*),\ T \mapsto T^\dagger) \quad \text{where } T^\dagger u^* := (Tu)^* \quad (u \in U),$$

defines a conjugate space for $L(U; V)$.

**Warning**

If $S \in L(U; V)$ and $T \in L(V; W)$ for complex vector spaces $U$, $V$ and $W$ with conjugations, then

$$(TS)^\dagger = T^\dagger S^\dagger,$$

which may not be what you expected. In particular, † is *not* an algebra involution (see below) on $L(V)$ unless $\dim V \leqslant 1$.

## LA-3

- Vector-valued function spaces $\mathcal{F}(S;V)$ and $\mathcal{F}_{00}(S;V)$
- $\mathcal{F}_{\mathbb{K}}(S)$ as dual of $\mathcal{F}_{00}(S;\mathbb{K})$
- Tensor product of functions
- Functorial properties
- Direct products and sums

### Definition

Let $S$ be a set and $V$ a vector space over $\mathbb{K}$. The *support* of $f \in \mathcal{F}(S;V)$ is

$$\operatorname{supp} f := \Big\{ s \in S : f(s) \neq 0 \Big\} = f^{-1}\Big( V \setminus \{0\} \Big).$$

The collection of functions $S \to V$ having finite support is denoted

$$\mathcal{F}_{00}(S;V) := \Big\{ f \in \mathcal{F}(S;V) : \operatorname{supp} f \subset\subset S \Big\},$$

### Notational abbreviations

$$\mathcal{F}_{\mathbb{K}}(S) := \mathcal{F}(S;\mathbb{K}), \quad \mathcal{F}(S) := \mathcal{F}(S;\mathbb{C}) \quad \text{and} \quad \mathcal{F}_{00}(S) := \mathcal{F}_{00}(S;\mathbb{C}).$$

### Example

For $a \in S$ and $v \in V$, a function $v\delta_a : S \to V$ is defined by

$$(v\delta_a)(s) = \delta_a(s)v := \begin{cases} v & \text{if } s = a, \\ 0 & \text{if } s \neq a. \end{cases}$$

Since $\operatorname{supp}(v\delta_a) = \{a\}$, $v\delta_a \in \mathcal{F}_{00}(S;V)$.

### Proposition (Spaces of functions)

*Let $S$ be a set and let $V$ be a vector space over $\mathbb{K}$. With addition and scalar multiplication defined pointwise:*

$$(f+g)(s) := f(s) + g(s) \quad \textit{and} \quad (\lambda f)(s) := \lambda f(s) \quad (s \in S),$$

(a) *$\mathcal{F}(S; V)$ is a vector space over $\mathbb{K}$, whose zero is the constant function 0;*

(b) *$\mathcal{F}_{00}(S; V)$ is a subspace of $\mathcal{F}(S; V)$;*

(c) *for each $f \in \mathcal{F}_{00}(S; V)$,*

$$f = \sum_{s \in S} f(s)\delta_s.$$

### Proof.

Straightforward verification (exercise). (Note that the sum is finite.) □

### Remark

Additional structure on $V$, such as that of an algebra, is inherited by $\mathcal{F}(S; V)$. Thus, for example, if $J$ is an ideal of an algebra $A$ then $\mathcal{F}_{00}(S; J)$ is an ideal of the algebra $\mathcal{F}(S; A)$.

# Natural realisation of $\mathcal{F}_{00}(S;\mathbb{K})^{\mathrm{dual}}$ — what about $\mathcal{F}_{00}(S;V)^{\mathrm{dual}}$?

**Proposition ($\mathcal{F}_{\mathbb{K}}(S)$ as a dual vector space)**

*Let $S$ be a set. Then, with $\delta$ denoting the map $S \to \mathcal{F}_{00}(S;\mathbb{K})$, $s \mapsto \delta_s$,*

$$\mathcal{F}_{00}(S;\mathbb{K})^{\mathrm{dual}} \to \mathcal{F}_{\mathbb{K}}(S), \quad \varphi \mapsto \varphi \circ \delta$$

*defines an isomorphism, with inverse*

$$f \mapsto \varphi_f \quad \textit{where } \varphi_f(g) := \sum_{s \in S} f(s)g(s)$$

**Proof.**

Straightforward verification. □

**Remark**

Note that the above sum is finite since

$$\operatorname{supp}(fg) \subset \operatorname{supp} g \subset\subset S.$$

## Notation

Let $S$ and $T$ be sets. For $f \in \mathcal{F}_{\mathbb{K}}(S)$ and $g \in \mathcal{F}_{\mathbb{K}}(T)$, define

$$f \otimes g : S \times T \to \mathbb{K}, \quad (s,t) \mapsto f(s)g(t).$$

## Remarks

- The map $(f,g) \mapsto f \otimes g$ is bilinear $\mathcal{F}_{\mathbb{K}}(S) \times \mathcal{F}_{\mathbb{K}}(T) \to \mathcal{F}_{\mathbb{K}}(S \times T)$.
- For $f \in \mathcal{F}_{\mathbb{K}}(S)$ and $g \in \mathcal{F}_{\mathbb{K}}(T)$, $\operatorname{supp}(f \otimes g) = \operatorname{supp} f \times \operatorname{supp} g$.
- For $s \in S$ and $t \in T$, $\delta_s \otimes \delta_t = \delta_{(s,t)}$.
- $\operatorname{Lin}\left\{f \otimes g : f \in \mathcal{F}_{00}(S;\mathbb{K}),\ g \in \mathcal{F}_{00}(S;\mathbb{K})\right\} = \mathcal{F}_{00}(S \times T;\mathbb{K})$.

## Question

What if $f \in \mathcal{F}(S;U)$ and $g \in \mathcal{F}(T;V)$ for vector spaces $U$ and $V$ over $\mathbb{K}$?

## $\mathcal{F}_{\mathbb{K}}(S \times T) \setminus \mathcal{F}_{\mathbb{K}}(S) \otimes \mathcal{F}_{\mathbb{K}}(T)$

Example

Let $F \in \mathcal{F}(S;T)$ for sets $S$ and $T$, and set

$$G = \text{graph}(F) := \Big\{(s, F(s)) : s \in S\Big\} \subset S \times T.$$

If $S$ is infinite and $F$ is injective then the *indicator function*

$$1_G : S \times T \to \mathbb{R}, (s,t) \mapsto \begin{cases} 1 & \text{if } t = F(s) \\ 0 & \text{if not} \end{cases}$$

is *not* (a sum of functions) of the form $f \otimes g$. Thus

$$1_G \notin \mathcal{F}_{\mathbb{K}}(S) \otimes \mathcal{F}_{\mathbb{K}}(T) := \text{Lin}\{f \otimes g : f \in \mathcal{F}_{\mathbb{K}}(S), g \in \mathcal{F}_{\mathbb{K}}(T)\}$$

Exercise. Prove this and find another (simpler?) example.

# $\mathcal{F}(S;V)$ and $\mathcal{F}_{00}(S;V)$ – basic functorial properties

## Proposition

*For sets $S$ and $T$, vector spaces $V$ and $W$ over $\mathbb{K}$, subspace $U$ of $V$ and $n \in \mathbb{N}$:*

- $\mathcal{F}(\emptyset;V) = \mathcal{F}_{00}(\emptyset;V) = \{0\}$ *(by convention; $\mathcal{F}(S;\{0\}) = \{0\}$);*
- $\mathcal{F}_{00}(S;V) \underset{\neq}{\subset} \mathcal{F}(S;V)$ *unless $S$ is finite or $V = \{0\}$;*

*and there are natural isomorphisms:*

- $\mathcal{F}(\{1,\dots,n\};V) \cong V^n := V \oplus \cdots \oplus V$ *($n$-fold direct sum);*
- $\mathcal{F}_{\mathbb{K}}(S \cup T) \cong \mathcal{F}_{\mathbb{K}}(S) \oplus \mathcal{F}_{\mathbb{K}}(T)$ *(assuming $S$ and $T$ disjoint);*
- $\mathcal{F}(S;V \oplus W) \cong \mathcal{F}(S;V) \oplus \mathcal{F}(S;W)$*;*
- $\mathcal{F}(S;V/U) \cong \mathcal{F}(S;V)/\mathcal{F}(S;U)$*;*
- $\mathcal{F}(S \times T;V) \cong \mathcal{F}(S;\mathcal{F}(T;V))$*;*

*and corresponding isomorphisms in which $\mathcal{F}$ is replaced by $\mathcal{F}_{00}$.*

## Proof.

Straightforward verifications of intelligent guesses! (Exercise.) □

### Definition (Direct sums and products)

Let $(V_s)_{s\in S}$ be an indexed family of vector spaces over $\mathbb{K}$. Then the set $\prod_{s\in S} V_s$ is endowed with vector space structure by componentwise operations:

$$(v_s)_{s\in S} + (v_s')_{s\in S} := (v_s + v_s')_{s\in S} \quad \text{and} \quad \lambda(v_s)_{s\in S} := (\lambda v_s)_{s\in S}.$$

The resulting vector space over $\mathbb{K}$ is called the (*external*) *direct product* of $(V_s)_{s\in S}$. The following subspace is called the (*external*) *direct sum* of $(V_s)_{s\in S}$ and is denoted $\sum_{s\in S}^{\oplus} V_s$:

$$\left\{ (v_s)_{s\in S} \;\middle|\; \operatorname{supp} v \subset\subset S \right\} \text{ where } \operatorname{supp} v := \{s \in S : v_s \neq 0\}.$$

### Remark

$\mathcal{F}(S;V)$ and $\mathcal{F}_{00}(S;V)$ are the special cases in which each $V_s$ equals the vector space $V$.

# LA-4

- Linear span; linear independence; basis; indexed basis
- Existence of (Hamel) bases
- Four characterisations of indexed bases
- Principle of Linear Extension
- Isomorphism $V^{\mathrm{dual}} \cong \mathcal{F}_{\mathbb{K}}(S)$ induced by basis choice for $V$
- Dimension
- Nonisomorphism of $V$, $V^{\mathrm{dual}}$, $(V^{\mathrm{dual}})^{\mathrm{dual}}, \ldots$ when $\dim V = \infty$

# Linear independence, spanning and basis – definition and example

### Definition

Let $E \subset V$ for a vector space $V$ over $\mathbb{K}$. With the convention that $\operatorname{Lin} \emptyset = \{0\}$, its *linear span* may be expressed as follows

$$\operatorname{Lin} E = \left\{ \sum_{e \in E} \mu(e)e : \mu \in \mathcal{F}_{00}(E; \mathbb{K}) \right\}$$

- $E$ *(linearly) spans* $V$ if $\operatorname{Lin} E = V$;
- $E$ is *linearly independent* if, for all $\mu \in \mathcal{F}_{00}(E; \mathbb{K})$ nonzero, $\sum_{e \in E} \mu(e)e \neq 0$;
- $E$ is a *basis* for $V$ if it is linearly independent and spans $V$;
- an indexed family $(e_s)_{s \in S}$ in $V$ is an *indexed basis for* $V$ if the map $e : S \to V$ is injective and its image is a basis.

A basis $E$ is easily rendered an indexed basis: $(e)_{e \in E}$.

### Example

Let $B$ be a basis for a vector space $V$. Then an indexed basis for $\mathcal{F}_{00}(S; V)$ is given by

$$(e\delta_s)_{(s,e) \in S \times B}.$$

### Theorem (Existence of bases)

*Let $V$ be a vector space over $\mathbb{K}$, let $L, S \subset V$ and suppose that*

- *$L$ is linearly independent,*
- *$L \subset S$, and*
- *$S$ spans $V$.*

*Then there is a basis $B$ for $V$ satisfying*

$$L \subset B \subset S.$$

### Proof.

The collection $\mathscr{X}$ of linearly independent sets $T$ satisfying $L \subset T \subset S$ is nonempty (since it includes $L$) and partially ordered by inclusion. It is easily verified that Zorn's Lemma applies to $\mathscr{X}$ and that a maximal element is a basis for $V$. □

### Remark

That every vector space $V$ has a basis now follows since we may take $L = \emptyset$ and $S = V$.

## Proposition (Characterisations of indexed bases)

*Let $(e_s)_{s\in S}$ be an indexed set in a vector space $V$ over $\mathbb{K}$. Then TFAE:*

(i) $(e_s)_{s\in S}$ *is an indexed basis for* $V$;

(ii) $\forall_{v\in V}\ \exists!_{\mu\in\mathcal{F}_{00}(S;\mathbb{K})} : v = \sum_{s\in S}\mu(s)e_s$ *(Basis Expansion);*

(iii) $\forall_{f\in\mathcal{F}_{\mathbb{K}}(S)}\ \exists!_{\varphi\in V^{\mathrm{dual}}}\ \forall_{s\in S}\ \varphi(e_s) = f(s)$;

(iv) *there is a unique isomorphism* $\Delta = \Delta^e : V \to \mathcal{F}_{00}(S;\mathbb{K})$ *satisfying* $\Delta e_s = \delta_s$ *for all* $s\in S$ *(Induced Isomorphism);*

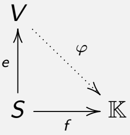
(v) *for any vector space* $W$ *over* $\mathbb{K}$ *and map* $\alpha\in\mathcal{F}(S;W)$ *there is a unique map* $A\in L(V;W)$ *satisfying* $Ae_s = \alpha(s)$ *for all* $s\in S$ *(Universal Property).*

## Proof.

A nice exercise, which may be *extended* by dividing the above into four characterisations of linear independence and four characterisations of linear spanning. □

Commutative diagrams for 3 of the 4 characterisations

(iii)

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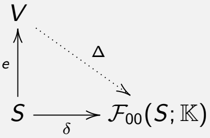


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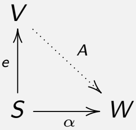


**Corollary (Principle of Linear Extension)**

*Let $B \subset V$ for a vector space $V$ over $\mathbb{K}$. Then TFAE:*

(i) *$B$ is a basis for $V$;*

(ii) $\forall_{v \in V} \exists!_{\mu \in \mathcal{F}_{00}(B;\mathbb{K})}$: $v = \sum_{e \in B} \mu(e) e$;

(iii) *for any $f \in \mathcal{F}_{\mathbb{K}}(B)$ there is a unique $\varphi \in V^{\mathrm{dual}}$ extending $f$;*

(iv) *the map $\delta : B \to \mathcal{F}_{00}(B;\mathbb{K})$, $e \mapsto \delta_e$, extends to an isomorphism $\Delta : V \to \mathcal{F}_{00}(B;\mathbb{K})$;*

(v) *for any vector space $W$ over $\mathbb{K}$ and map $\tau \in \mathcal{F}(B;W)$ there is a unique $T \in L(V;W)$ extending $\tau$.*

Corollary

*Let $V$ be a vector space over $\mathbb{K}$. For any indexed basis $(e_s)_{s\in S}$ for $V$, the map*

$$V^{\mathrm{dual}} \to \mathcal{F}_{\mathbb{K}}(S), \quad \varphi \mapsto \varphi \circ e$$

*is an isomorphism ($e$ here standing for the function $s \mapsto e_s$).*
*More generally, for any other vector space $W$ over $\mathbb{K}$, the map*

$$L(V; W) \to \mathcal{F}(S; W), \quad T \mapsto T \circ e$$

*defines an isomorphism.*

**Theorem (Dimension as Complete Invariant)**

*Let $B$ and $B'$ be bases respectively for vector spaces $V$ and $V'$ over $\mathbb{K}$. Then TFAE:*

(i) $V' \cong V$;

(ii) $\operatorname{card} B' = \operatorname{card} B$.

**Proof.**

For any bijection $\gamma : B \to B'$, the induced map $\mathcal{F}(B'; \mathbb{K}) \to \mathcal{F}(B; \mathbb{K})$, $f \mapsto f \circ \gamma$, is easily seen to be an isomorphism. The implication (ii) $\implies$ (i) therefore follows from part (iv) of the Proposition.

Conversely, suppose that $V' \cong V$. Since an isomorphic image of a basis is a basis (by part (ii) of the Proposition) we may assume without loss that $V' = V$. Moreover, by symmetry it suffices to show that $\operatorname{card} B \leqslant \operatorname{card} B'$.

Finally, the result is well known to you when $V$ or $V'$ is finite dimensional, so we may assume that $B$ and $B'$ are infinite. For each $e' \in B'$ let $\sum_{e \in B} \mu_{e'}(e)e$ be its basis expansion with respect to $B$. Then $\bigcup_{e' \in B'} \operatorname{supp} \mu_{e'} = B$ (verify!), and it follows by cardinal arithmetic that $\operatorname{card} B \leqslant \aleph_0 \operatorname{card} B' = \operatorname{card} B'$. ☐

### Definition (Dimension)

Let $V$ be a vector space over $\mathbb{K}$. The *dimension* of $V$, denoted by $\dim V$, is the cardinality of any basis for $V$.

### Proposition (Jacobson)

*Let $V$ be an infinite-dimensional vector space over $\mathbb{K}$ $(= \mathbb{R}$ or $\mathbb{C})$. Then:*

(a) $\operatorname{card} V = \mathfrak{c} \cdot \dim V$

(b) $\dim V^{\mathrm{dual}} = 2^{\dim V}$

*where $\mathfrak{c}$ is the cardinality of the continuum.*

### Proof.

Exercise.
[Hints: $\mathcal{F}_{00}(S; \mathbb{K}) = \bigcup_{A \subset\subset S} \left\{ f \in \mathcal{F}_{00}(S; \mathbb{K}) : \operatorname{supp} f \subset A \right\}$ and $c = 2^{\aleph_0}$]

### Corollary

*For an infinite-dimensional vector space $V$ over $\mathbb{K}$ $(= \mathbb{R}$ or $\mathbb{C})$, the vector spaces $V$, $V^{\mathrm{dual}}$, $(V^{\mathrm{dual}})^{\mathrm{dual}}, \ldots$ are all nonisomorphic.*

□

## LA-5

- Complementary subspaces
- Linear extension from subspaces
- $V^{\mathrm{dual}}$ separates the elements of $V$
- Linear bidual embedding
- Transpose of a linear map

### Proposition (Left-inverse for subspace inclusions)

*Let $V_0$ be a subspace of a vector space $V$ over $\mathbb{K}$. Then the inclusion map $J : V_0 \to V$ has a linear left-inverse $R$ whose kernel is a complementary subspace to $V_0$, that is*

$$V_0 \cap \operatorname{Ker} R = \{0\} \quad \textit{and} \quad V_0 + \operatorname{Ker} R = V.$$

### Proof.

Exercise. [Hint: Let $B$ be a basis for $V$ which includes a basis $B_0$ for $V_0$. Define a map $\tau : B \to V_0$ by $\tau(e) = e$ if $e \in B_0$, $\tau(e) = 0$ otherwise. Let $R$ be the unique linear extension of $\tau$.] □

### Remark

$P := JR \in L(V)$ is the *projection onto $V_0$ along $V_1$*, where $V_1 = \operatorname{Ker} R$. [In the sketch proof above,

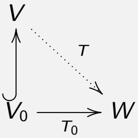
$$V_1 = \operatorname{Ker} R = \operatorname{Lin}(B \setminus B_0).]$$

## Corollary (Linear extension from subspaces)

*Let $T_0 \in L(V_0; W)$ for a vector space $W$ over $\mathbb{K}$ and a subspace $V_0$ of a vector space $V$ over $\mathbb{K}$. Then $T_0$ has an extension $T \in L(V; W)$.*

## Proof.

Set $T = T_0 R$ where $R$ is a left-inverse of the inclusion map $V_0 \to V$. □



**Corollary ($V^{\mathrm{dual}}$ separates the elements of $V$)**

*Let $V$ and $W$ be vector spaces over $\mathbb{K}$ with $\dim W \geqslant 1$ (e.g. $W = \mathbb{K}$). Then, for $v \in V$, TFAE:*

(i) $Tv = 0$ *for all* $T \in L(V; W)$;

(ii) $v = 0$.

**Proof.**

Exercise. $\square$

**Proposition (Linear bidual embedding)**

*Let $V$ be a vector space over $\mathbb{K}$. Then the map $J_V : V \to (V^{\mathrm{dual}})^{\mathrm{dual}}$ given by*

$$v \mapsto \hat{v}, \quad \text{where } \hat{v}(\varphi) := \varphi(v) \quad (\varphi \in V^{\mathrm{dual}}),$$

*is a linear injection, called the* linear bidual embedding *of $V$.*

**Proof.**

Exercise. [Direct verification and appeal to the previous corollary.] □

**Remark**

By dimension arguments TFAE:

- $J_V$ is surjective, and thus is an isomorphism;
- $V$ is finite dimensional.

# Transpose of a linear map

## Definition

Let $A \in L(U; V)$ for vector spaces $U$ and $V$ over $\mathbb{K}$. Its *transpose* $A^\top \in L(V^{\mathrm{dual}}; U^{\mathrm{dual}})$ is given by

$$A^\top \psi = \psi \circ A.$$

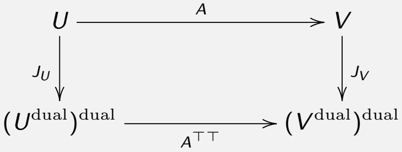
## Proposition (Consistency of transpose with bidual embedding)

*Let $A \in L(U; V)$ for vector spaces $U$ and $V$ over $\mathbb{K}$. Then:*

$$A^{\top\top} \hat{u} = \widehat{Au} \quad \textit{for } u \in U.$$

## Proof.

Exercise. ☐



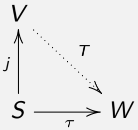
## LA-6 (for self-study)

- Free vector space on a set over $\mathbb{K}$

# Free vector space on a set $S$ over $\mathbb{K}$

## Definition (Free vector space on $S$ over $\mathbb{K}$)

Let $S$ be a set. A pair $(V, j)$, consisting of a vector space $V$ over $\mathbb{K}$ and a map $j : S \to V$, is a *free vector space on* $S$ *over* $\mathbb{K}$ if it enjoys the following *Universal Property*: For every pair $(W, \tau)$ consisting of a vector space $W$ over $\mathbb{K}$ and map $\tau : S \to W$ there is a unique $T \in L(V; W)$ such that $T \circ j = \tau$:




## Remark

One says that $(V, j)$ "linearises" the set $S$ and every vector-valued map from $S$.

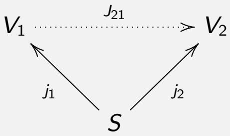
# Existence and uniqueness of free vector spaces

**Theorem (Existence and uniqueness of free vector spaces)**

*Let $S$ be a set. Then:*

(a) *There is a free vector space on $S$ over $\mathbb{K}$.*

(b) *If $(j_1, V_1)$ and $(j_2, V_2)$ are free vector spaces on $S$ over $\mathbb{K}$ then there is a unique linear isomorphism $J_{21} : V_1 \to V_2$ satisfying $J_{21} \circ j_1 = j_2$:*



Proof.

(a) Let $\delta$ be the map $S \to \mathcal{F}_{00}(S;\mathbb{K})$, $s \mapsto \delta_s$. Then $(\mathcal{F}_{00}(S;\mathbb{K}), \delta)$ is a free vector space on $S$ over $\mathbb{K}$ by the Universal Property of Indexed Bases, applied to $(\delta_s)_{s\in S}$.

(b) Repeated application of the universal property of free vector spaces on $S$ gives unique maps $J_{21} \in L(V_1; V_2)$, $J_{12} \in L(V_2; V_1)$, $I_1 \in L(V_1)$ and $I_2 \in L(V_2)$ satisfying

$$J_{21} \circ j_1 = j_2, \quad J_{12} \circ j_2 = j_1, \quad I_1 \circ j_1 = j_1 \quad \text{and} \quad I_2 \circ j_2 = j_2.$$

Clearly $I_1 = I_{V_1}$ and $I_2 = I_{V_2}$. Since $J_{12}J_{21} \circ j_1 = J_{12} \circ j_2 = j_1$ it follows that $J_{12}J_{21} = I_1 = I_{V_1}$. Similarly $J_{21}J_{12} = I_{V_2}$, so $J_{21}$ is an isomorphism.

□

Terminology and notation

"The" free vector space on $S$ over $\mathbb{K}$; we shall denote it $\mathrm{F}_{\mathbb{K}}(S)$.

## LA-7

- Algebras
- Spectrum of a unital algebra element
- Commutation and spectrum
- Left regular representation of an algebra

### Definition

An (associative) *algebra* $\mathcal{A}$ over $\mathbb{K}$ is a vector space over $\mathbb{K}$ together with a *product* $m : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$, written $(a, a') \mapsto a \cdot a'$, which is bilinear and associative. The algebra is *involutive* if it has a linear involution satisfying

$$(ab)^* = b^* a^* \text{ for all } a, b \in \mathcal{A};$$

it is *unital* if it has a multiplicative identity (commonly written $e$ or 1). The *group of units* of a unital algebra $\mathcal{A}$, denoted $GL(\mathcal{A})$, or $\mathcal{A}^\times$, is the group

$$\{a \in \mathcal{A} : a \text{ has a multiplicative inverse}\}.$$

The *spectrum* of an element $a$ of a complex unital algebra $\mathcal{A}$ is the set

$$\sigma(a) := \{\lambda \in \mathbb{C} : (\lambda e - a) \notin GL(\mathcal{A})\}.$$

### Remarks

- An algebra homomorphism (linear, multiplicative map) of unital algebras *need not be* unital (preserve identities). Exercise: Find an example.
- $\sigma(\mu e + \nu a) = \{\mu\} + \nu \sigma(a)$, for $\mu, \nu \in \mathbb{C}$, $a \in \mathcal{A}$.

## Basic examples

- $M_n(\mathbb{C})$ is an involutive unital algebra, with involution given by the matrix-adjoint $(a^*)_{ij} := (a_{ji})^*$ (for $i, j = 1, \ldots, n$) and group of units $GL(M_n(\mathbb{C})) = GL_n(\mathbb{C})$.
  For $a \in M_n(\mathbb{C})$, $\sigma(a) = \{\lambda \in \mathbb{C} : \lambda \text{ is an eigenvalue of } a\}$.
  $M_n(\mathbb{C})$ is a simple algebra, that is it has no proper nontrivial ideals (exercise).
- $\mathcal{F}(S)$ is an involutive unital algebra, with involution given by pointwise complex conjugation $f^*(s) := f(s)^*$ ($s \in S$) and group of units

$$GL(\mathcal{F}(S)) = \{f \in \mathcal{F}(S) : \operatorname{supp} f = S\}.$$

  For $f \in \mathcal{F}(S)$, $\sigma(f) = \operatorname{Ran} f := f(S)$.
  $\mathcal{F}_{00}(S)$ is a proper ideal, unless $S$ is finite.

## Remark

These generalise by replacing $M_n(\mathbb{C})$ by $M_n(\mathcal{A})$ and $\mathcal{F}(S)$ by $\mathcal{F}(S; \mathcal{A})$, for an (involutive and/or unital) algebra $\mathcal{A}$.
For example, $M_n(\mathcal{A})$ is a simple algebra if and only if $\mathcal{A}$ is.

**Proposition**

*Let $\mathcal{A}$ be a unital algebra and let $a, b \in \mathcal{A}$. If $(e - ab) \in GL(\mathcal{A})$ then $(e - ba) \in GL(\mathcal{A})$ too, and*

$$(e - ba)^{-1} = e + b(e - ab)^{-1}a.$$

**Proof.**

Straightforward verification (exercise). □

**Corollary (Quasi-commutativity of the spectrum)**

*Let $\mathcal{A}$ be a complex unital algebra and let $a, b \in \mathcal{A}$. Then*

$$\sigma(ba) \setminus \{0\} = \sigma(ab) \setminus \{0\}.$$

**Proof.**

Exercise. Further, give examples in which $\sigma(ba) \neq \sigma(ab)$. □

Proposition (Spectral implication of commutation relations)

*Let $\mathcal{A}$ be a complex unital algebra and let $a, b \in \mathcal{A}$ satisfy the commutation relations*

$$ba - ab = \mu e$$

*where $\mu \in \mathbb{C} \setminus \{0\}$. If $\sigma(ab)$ is nonempty then it is unbounded.*

Proof.

Exercise. [Hint: For $\lambda \in \sigma(ab)$ show that $\{\lambda\} + \mu\mathbb{Z}_+ \subset \sigma(ab)$ if $\lambda \notin -\mu\mathbb{Z}_+$ and $\{\lambda\} - \mu\mathbb{N} \subset \sigma(ab)$ if $\lambda \notin \mu\mathbb{N}$.] □

# Left regular representation of an algebra

**Definition**

An algebra $\mathcal{A}$ is *nondegenerate* if it satisfies the following condition:

$$x \in \mathcal{A}, \quad x\mathcal{A} = \{0\} \quad \implies \quad x = 0.$$

**Proposition**

*Let $\mathcal{A}$ be an algebra over $\mathbb{K}$. The map*

$$\lambda : \mathcal{A} \to L(\mathcal{A}), \quad x \mapsto L_x, \quad \text{where } L_x a := xa \quad (a \in \mathcal{A}),$$

*is an algebra homomorphism.*
*Moreover, $\lambda$ is unital if $\mathcal{A}$ is unital, and $\lambda$ is injective if $\mathcal{A}$ is nondegenerate.*

**Proof.**

Straightforward verification. □

**Remark**

This provides a means of embedding a nonunital algebra into a unital algebra.

## LA-8

- Finite rank linear maps

## Notation

Let $V$ be a vector space $\mathbb{K}$. For $v \in V$ define

$$|v\rangle \in L(\mathbb{K}; V) \quad \text{by} \quad |v\rangle\lambda := \lambda v \quad (\lambda \in \mathbb{K}).$$

## Remarks

- The map $v \mapsto |v\rangle$ is an isomorphism $V \to L(\mathbb{K}; V)$.
- For a second vector space $U$ over $\mathbb{K}$ and elements $\varphi \in U^{\mathrm{dual}}$ and $u \in U$, we have
$$|v\rangle\varphi \in L(U; V) \quad \text{and} \quad |v\rangle\hat{u} \in L(U^{\mathrm{dual}}; V).$$

## Definition

Let $T \in L(U; V)$ for vector spaces $U$ and $V$ over $\mathbb{K}$. Then the *rank* of $T$ is defined by

$$\operatorname{rank} T := \dim(\operatorname{Ran} T).$$

We write $L_{00}(U; V)$ for the collection of finite rank linear maps $U \to V$.

### Proposition (Representation of finite rank maps)

*Let $T \in L_{00}(V;W)$ for vector spaces $V$ and $W$ over $\mathbb{K}$. Then, for any indexed basis $(e_j)_{j\in K}$ for* Ran $T$, *there is an indexed family $(\psi_j)_{j\in K}$ in $V^{\mathrm{dual}}$ such that*

$$T = \sum_{j\in K} |e_j\rangle\psi_j \qquad (*)$$

### Proof.

Let $J\widetilde{T}Q$ be the canonical factorisation of $T$. Then it is easily checked that

$$J = \sum_{j\in K} |e_j\rangle\varphi_j$$

where $(\varphi_j)_{j\in K}$ is the dual basis to $(e_j)_{j\in K}$ (determined by $\varphi_j(e_k) = \delta_{jk}$ $(j,k \in K)$). Thus (*) follows with $\psi_j := \varphi_j \circ \widetilde{T}Q$. □

**Proposition**

*Let $U$, $V$ and $W$ be vector spaces over $\mathbb{K}$.*

(a)
$$L_{00}(V;W)L(U;V) \subset L_{00}(U;W)$$
*and*
$$L(V;W)L_{00}(U;V) \subset L_{00}(U;W).$$
*In particular, $L_{00}(V)$ is an ideal of the unital algebra $L(V)$.*

(b)
$$(|v\rangle\varphi)^{\top} = |\varphi\rangle\hat{v} \quad \text{for } v \in V \text{ and } \varphi \in U^{\mathrm{dual}}.$$
*In particular $A^{\top} \in L_{00}(V^{\mathrm{dual}}; U^{\mathrm{dual}})$ if $A \in L_{00}(U;V)$.*

(c) *Let $A \in L(U;V)$. If $A^{\top} \in L_{00}(V^{\mathrm{dual}}; U^{\mathrm{dual}})$, then $A \in L_{00}(U;V)$.*

**Proof.**

Exercise. Show also that either inclusion in (a) can be proper. □

# LA-9

- Multilinear maps
- Linear tensor products
- Basis characterisations of tensor products
- Existence and uniqueness of linear tensor products
- Models and realisations of tensor products
- Trace on $L_{00}(V)$
- Tensor products of linear maps
- Higher-order tensor products
- Tensor products of algebras

### Definition (Multilinear maps)

Let $U, U_1, \ldots, U_n$ and $W$ be vector spaces over $\mathbb{K}$. A map

$$\alpha : U_1 \times \cdots \times U_n \to W$$

is *multilinear* if it is linear in each of its arguments (the other elements being held fixed).

Write $ML(U_1, \ldots, U_n; W)$ for the collection of these, and $ML_{\mathrm{sym}}(U^{\times n}; W)$ for the collection of *symmetric* multilinear maps $U \times \cdots \times U \to W$ (satisfying $\alpha \circ \pi_\sigma^{-1} = \alpha$ for all $\sigma \in \mathscr{S}_n$, where $\pi_\sigma(u) := (u_{\sigma(1)}, \ldots, u_{\sigma(n)})$).

### Remarks

- $ML(U_1, \ldots, U_n; W)$ is a subspace of $\mathcal{F}(U_1 \times \cdots \times U_n; W)$ and
- $ML_{\mathrm{sym}}(U^{\times n}; W)$ is a subspace of $\mathcal{F}_{\mathrm{sym}}(U^{\times n}; W)$.

where $\mathcal{F}_{\mathrm{sym}}$ denotes the class of symmetric functions.

- There are natural isomorphisms

$$ML(U_1, \ldots, U_n; W) \cong ML\big(U_1, \ldots, U_{n-k}; ML(U_{n-k+1}, \ldots, U_n; W)\big)$$

and

$$ML(U_1, \ldots, U_n; W) \cong ML(U_{\sigma(1)}, \ldots, U_{\sigma(n)}; W), \quad \text{for } \sigma \in \mathscr{S}_n;$$

given by restrictions of the isomorphisms

$$\mathcal{F}(U_1 \times \cdots \times U_n; W) \to \mathcal{F}(U_1 \times \cdots \times U_{n-k}; \mathcal{F}(U_{n-k+1} \times \cdots \times U_n; W)),$$

respectively,

$$\mathcal{F}(U_1 \times \cdots \times U_n; W) \to \mathcal{F}(U_{\sigma(1)} \times \cdots \times U_{\sigma(n)}; W) :$$
$$\widetilde{f}(u_1, \ldots, u_{n-k})(u_{n-k+1}, \ldots, u_n) := f(u), \quad \text{resp. } \widetilde{f} = f \circ \pi_\sigma^{-1}.$$

- Specialising, we have:

$$ML(U, V; \mathbb{K}) \cong L(U; V^{\mathrm{dual}}) \text{ and } ML(U, V; W) \cong ML(V, U; W).$$

## Definition (Linear tensor products)

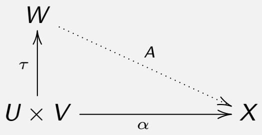
Let $U$ and $V$ be vector spaces over $\mathbb{K}$. A *tensor product* of the ordered pair $(U, V)$ is a vector space $W$ over $\mathbb{K}$ together with a bilinear map

$$\tau : U \times V \to W$$

enjoying the following universal property: for any vector space $X$ over $\mathbb{K}$ and $\alpha \in ML(U, V; X)$ there is a unique linear map $A : W \to X$ satisfying $A \circ \tau = \alpha$.

## Remarks

- We say that $(W, \tau)$ "linearises" every bilinear (vector-valued) map from $U \times V$.
- The universal property is captured by the commutative diagram:



Proposition (Basis Characterisations of Tensor Products)

*Let $\tau \in ML(U, V; W)$ for vector spaces $U$, $V$ and $W$ over $\mathbb{K}$.*
*Then TFAE:*

(i) $(W, \tau)$ *is a tensor product of* $(U, V)$*;*

(ii) *for any* $\mu \in ML(U, V; \mathbb{K})$ *there is a unique* $\varphi \in W^{\mathrm{dual}}$ *such that* $\varphi \circ \tau = \mu$*;*

(iii) *for any bases* $B_1$ *for* $U$ *and* $B_2$ *for* $V$, $(\tau(e, f))_{(e,f) \in B_1 \times B_2}$ *is an indexed basis for* $W$*;*

(iv) *there are bases* $B_1$ *for* $U$ *and* $B_2$ *for* $V$ *such that* $(\tau(e, f))_{(e,f) \in B_1 \times B_2}$ *is an indexed basis for* $W$*.*

Exercise. Prove the following useful further equivalent.

(v) $W = \operatorname{Lin}\operatorname{Ran}\tau$ and $(\tau(e, f))_{(e,f) \in B_1 \times B_2}$ is linearly independent for some bases $B_1$ and $B_2$ for for $U$ and $V$ respectively.

Proof.

The implications (i) $\implies$ (ii) and (iii) $\implies$ (iv) are immediate.

(ii) $\implies$ (iii): Suppose that (ii) holds and $B_1$ is a basis for $U$ and $B_2$ is a basis for $V$. Let $F \in \mathcal{F}_{\mathbb{K}}(B_1 \times B_2)$. Since $B_1$ and $B_2$ are bases it follows that $F$ has a unique bilinear extension $\mu_F : U \times V \to \mathbb{K}$. Let $\varphi \in W^{\mathrm{dual}}$ be the unique linear function such that $\varphi \circ \tau = \mu_F$. Then, by bilinearity, $\varphi$ is the unique functional in $W^{\mathrm{dual}}$ satisfying $\varphi(\tau(e, f)) = F(e, f)$ for all $e \in B_1$, $f \in B_2$. By the Characterisation of Indexed Bases, this implies that $(\tau(e, f))_{e \in B_1,\ f \in B_2}$ is an indexed basis for $W$, so (iii) holds.

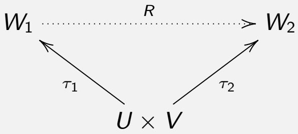
(iv) $\implies$ (i): Suppose that (iv) holds and $\alpha \in ML(U, V; X)$ for a vector space $X$ over $\mathbb{K}$. Let $B_1$ and $B_2$ be bases for $U$ and $V$, respectively, for which $(\tau(e, f))_{(e,f) \in B_1 \times B_2}$ is an indexed basis for $W$. By the Principle of Linear Extension there is a unique map $A \in L(W; X)$ satisfying $A\tau(e, f) = \alpha(e, f)$ for all $e \in B_1$, $f \in B_2$. By bilinearity, $A$ satisfies $A\tau(u, v) = \alpha(u, v)$ for all $u \in U$, $v \in V$. Thus (i) holds. □

# Existence and uniqueness of linear tensor products

**Theorem (Existence and uniqueness of linear tensor products)**

*Let $U$ and $V$ be vector spaces over $\mathbb{K}$.*

- *There is a tensor product of $(U, V)$.*
- *If $(W_1, \tau_1)$ and $(W_2, \tau_2)$ are tensor products of $(U, V)$ then there is a unique linear isomorphism $R : W_1 \to W_2$ satisfying $R \circ \tau_1 = \tau_2$:*



**Existence.**
Let $W = \mathcal{F}_{00}(B_1 \times B_2; \mathbb{K})$ where $B_1$ and $B_2$ are bases for $U$ and $V$, respectively, and let $\tau = U \times V \to W$ be the map given by

$$\tau(u, v) = \widetilde{u} \otimes \widetilde{v}$$

where $u \mapsto \widetilde{u}$ and $v \mapsto \widetilde{v}$ are the isomorphisms

$$U \to \mathcal{F}_{00}(B_1; \mathbb{K}) \text{ and } V \to \mathcal{F}_{00}(B_2; \mathbb{K})$$

induced by the choices of bases. Now note three things: $\tau$ is bilinear,

$$\tau(e, f) = \delta_e \otimes \delta_f = \delta_{(e,f)} \quad \text{for } e \in B_1,\ f \in B_2,$$

and $(\delta_{(e,f)})_{(e,f)\in B_1 \times B_2}$ is an indexed basis for $W$. Thus, by the Basis Characterisation of Tensor Products, it follows that $(W, \tau)$ is a tensor product of $(U, V)$. □

**Uniqueness.**

By the universal property of $(W_1, \tau_1)$ there is a unique map $R_{21} \in L(W_1; W_2)$ such that $R_{21} \circ \tau_1 = \tau_2$, and so it remains only to show that $R_{21}$ is bijective. Similarly there are unique maps $R_{12} \in L(W_2; W_1)$ and $I_1 \in L(W_1)$ such that $R_{12} \circ \tau_2 = \tau_1$ and $I_1 \circ \tau_1 = \tau_1$. Now $I_1$ must equal $I_{W_1}$ and

$$R_{12} R_{21} \circ \tau_1 = R_{12} \circ \tau_2 = \tau_1,$$

so $R_{12} R_{21} = I_1 = I_{W_1}$. By symmetry $R_{21} R_{12} = I_{W_2}$. Thus $R_{21}$ is an isomorphism. □

## Remarks

With existence and uniqueness of tensor products established, we refer to "the" tensor product of $(U, V)$ and write $U \otimes V$ for the space and $\otimes : (u, v) \mapsto u \otimes v$ for its bilinear map.

## Basic examples

- $\{0\} \otimes V = \{0\}$.
- $\mathbb{K} \otimes V = V$, $\lambda \otimes v := \lambda v$.
- $M_{n,m}(\mathbb{K}) \otimes M_{p,q}(\mathbb{K}) = M_{n,m}(M_{p,q}(\mathbb{K}))$, with

$$[a_{ij}] \otimes B := [a_{ij} B] \quad \text{(Kronecker product)}.$$

Examples

- Let $U_0$ and $V_0$ be subspaces of vector spaces $U$ and $V$ over $\mathbb{K}$. Then a natural realisation of $U_0 \otimes V_0$ is
$$\operatorname{Lin}\left\{u \otimes v : u \in U_0,\ v \in V_0\right\},$$
as a subspace of $U \otimes V$.
- For sets $S$ and $T$, $\mathcal{F}_{\mathbb{K}}(S) \otimes \mathcal{F}_{\mathbb{K}}(T)$ is naturally realised in $\mathcal{F}_{\mathbb{K}}(S \times T)$, with $f \otimes g$ having its standard meaning. (Exercise: Prove this.)

Remark

The induced realisation of $\mathcal{F}_{00}(S;\mathbb{K}) \otimes \mathcal{F}_{00}(T;\mathbb{K})$ as a subspace of $\mathcal{F}_{\mathbb{K}}(S \times T)$ is precisely $\mathcal{F}_{00}(S \times T;\mathbb{K})$. (This fact is implicitly revealed in our proof of existence of tensor products.)

# Popular models for/realisations of the linear tensor product

Let $U$ and $V$ be vector spaces over $\mathbb{K}$.

- $$U \otimes V := \operatorname{Lin}\{\widehat{u \times v} : u \in U,\ v \in V\} \subset ML(U^{\text{dual}}, V^{\text{dual}}; \mathbb{K}),$$
  with $u \otimes v = \widehat{u \times v}$ where
  $$\widehat{u \times v}(\varphi, \psi) := \varphi(u)\psi(v) \quad (u \in U,\ v \in V).$$
- $$U \otimes V := \operatorname{Lin}\{|v\rangle \hat{u} : u \in U,\ v \in V\} \subset L_{00}(U^{\text{dual}}; V),$$
  with $u \otimes v := |v\rangle \hat{u}$.
- $U \otimes V := \mathrm{F}_{\mathbb{K}}(U \times V)/N$, $u \otimes v := [(u, v)]$ where $N := \operatorname{Lin} E$ and
  $$E := \Big\{(u + \lambda u', v + \mu v') - (u, v) - \lambda(u', v) - \mu(u, v') - \lambda\mu(u', v') : u, u' \in U,\ v, v' \in V,\ \lambda, \mu \in \mathbb{K}\Big\}.$$

If one of the spaces is a dual space then there is a natural choice:

- $U^{\text{dual}} \otimes V := L_{00}(U, V)$, $\varphi \otimes v := |v\rangle\varphi$.

Exercise: Check these out.

# Trace on $L_{00}(V)$

## Definition (Trace on $L_{00}(V)$)

Let $V$ be a vector space over $\mathbb{K}$. In view of the realisation of $V^{\mathrm{dual}} \otimes V$ as $L_{00}(V)$, with $\varphi \otimes v := |v\rangle\varphi$ ($v \in V$, $\varphi \in V^{\mathrm{dual}}$), a linear functional – called the *trace on* $L_{00}(V)$ – is defined by linearisation of the bilinear functional

$$V^{\mathrm{dual}} \times V \to \mathbb{K}, \quad (\varphi, v) \mapsto \varphi(v).$$

- For any representation $\sum_{s\in S} |v_s\rangle\varphi_s$ (finite sum) for $A \in L_{00}(V)$,

$$\operatorname{Tr} A = \sum_{s\in S} \varphi_s(v_s).$$

- In particular,
  letting $(\widetilde{\varphi}_j)_{j\in J}$ be the dual basis of an indexed basis $(e_j)_{j\in J}$ for $\operatorname{Ran} A$,
  and setting $\varphi_j = \widetilde{\varphi}_j \circ R$ ($j \in J$)
  where $R$ is a left-inverse of the subspace inclusion $\operatorname{Ran} A \to V$,
  the map $\sum_{j\in J} |e_j\rangle\varphi_j \in L(V)$ is a projection with range $\operatorname{Ran} A$, so

$$A = \sum_{j\in J} |e_j\rangle(\varphi_j \circ A) \quad \text{and} \quad \operatorname{Tr} A = \sum_{j\in J} \varphi_j(Ae_j).$$

## Proposition (Tensor product of linear maps)

*Let* $U = U_1 \otimes U_2$ *and* $V = V_1 \otimes V_2$ *for vector spaces* $U_1, \dots, V_2$ *over* $\mathbb{K}$, *and let* $R_1 \in L(U_1; V_1)$ *and* $R_2 \in L(U_2; V_2)$.

(a) *There is unique map* $R \in L(U; V)$ *satisfying*

$$R(u_1 \otimes u_2) = R_1 u_1 \otimes R_2 u_2 \quad (u_1 \in U_1,\ u_2 \in U_2).$$

*Notation:* $R_1 \otimes R_2$.

(b) *If* $R_1$ *and* $R_2$ *are injective, then so is* $R$; *the converse holds provided that* $U \neq \{0\}$.

(c) *If* $R_1$ *and* $R_2$ *are surjective, then so is* $R$; *the converse holds provided that* $V \neq \{0\}$.

## Proof.

(a) follows since the map $(u_1, u_2) \mapsto R_1 u_1 \otimes R_2 u_2$ is bilinear;
(b) and (c) are left as exercises. □

# Realisations of the tensor products $L(U_1; V_1) \otimes L(U_2; V_2)$ and $L_{00}(U_1; V_1) \otimes L_{00}(U_2; V_2)$

**Proposition**

*Let $U = U_1 \otimes U_2$ and $V = V_1 \otimes V_2$ for vector spaces $U_1, \dots, V_2$ over $\mathbb{K}$.*

(a) *The bilinear map*

$$L(U_1; V_1) \times L(U_2; V_2) \mapsto L(U; V), \; (R_1, R_2) \mapsto R_1 \otimes R_2$$

*provides a realisation of $L(U_1; V_1) \otimes L(U_2; V_2)$ in $L(U; V)$.*

(b) *The induced realisation of $L_{00}(U_1; V_1) \otimes L_{00}(U_2; V_2)$ as a subspace of $L(U; V)$ is precisely $L_{00}(U; V)$ provided that $\dim U_1 < \infty$ or $\dim U_2 < \infty$.*

Proof.

Let $B_1$ be a basis for $L(U_1; V_1)$ and $B_2$ a basis for $L(U_2; V_2)$.
To be proved: the indexed set $(R \otimes S)_{(R,S) \in B_1 \times B_2}$ is linearly independent.
Suppose therefore that $\mu \in \mathcal{F}_{00}(B_1 \times B_2; \mathbb{K})$ and $T := \sum \mu(R,S) R \otimes S$ is the zero map; we must show that $\mu = 0$. For all $u_1 \in U_1$, $u_2 \in U_2$ and $\varphi \in V_2^{\mathrm{dual}}$,

$$\begin{aligned} 0 &= (\mathrm{id}_{V_1} \otimes \varphi) T(u_1 \otimes u_2) \\ &= \sum \mu(R,S) \varphi(S u_2) R u_1 = \left\{ \sum_{R \in B_1} \varphi\left( \sum_{S \in B_2} \mu(R,S) S u_2 \right) R \right\} u_1. \end{aligned}$$

Since (i) this holds for all $u_1 \in U_1$, (ii) $B_1$ is linearly independent, and (iii) $V_2^{\mathrm{dual}}$ separates $V_2$, this implies that, for all $R \in B_1$ and $u_2 \in U_2$,

$$0 = \sum_{S \in B_2} \mu(R,S) S u_2 = \left( \sum_{S \in B_2} \mu(R,S) S \right) u_2.$$

Thus, for all $R \in B_1$, $\sum_{S \in B_2} \mu(R,S) S = 0$. Therefore, by the linear independence of $B_2$, $\mu(R,S) = 0$ for all $R \in B_1$ and $S \in B_2$, in other words $\mu = 0$. This proves (a).
(b) is left as an exercise. □

### Definition

Let $U_1, \ldots, U_n$ be vector spaces over $\mathbb{K}$. A vector space $W$ over $\mathbb{K}$ with multilinear map $\tau : U_1 \times \cdots \times U_n \to W$ enjoying the Universal Property: for any vector space $X$ over $\mathbb{K}$ and multilinear map $\alpha : U_1 \times \cdots \times U_n \to X$ there is a unique $A \in L(W; X)$ such that $A \circ \tau = \alpha$, is called a *tensor product of* $(U_1, \ldots, U_n)$.

### Exercise

Show that such tensor products exist and are unique in the appropriate sense.

### Notation

$U_1 \otimes \cdots \otimes U_n$, $u_1 \otimes \cdots \otimes u_n$.

### Exercise

Show that the tensor product operation is associative in the following precise sense: (Let $U$, $V$ and $W$ be vector spaces over $\mathbb{K}$).
If $\tau : (U \otimes V) \times W \to U \otimes V \otimes W$ is the unique bilinear map satisfying

$$\tau(u \otimes v, w) = u \otimes v \otimes w \quad (u \in U,\ v \in V,\ w \in W)$$

then $(U \otimes V \otimes W, \tau)$ is a model for the tensor product of $(U \otimes V, W)$.

- Let $U$ and $V$ be vector spaces over $\mathbb{K}$. Then the map

$$V \times U \to U \otimes V, \quad (v, u) \mapsto u \otimes v$$

is bilinear and its linearisation is easily seen to be an isomorphism $V \otimes U \to U \otimes V$, called a *tensor flip*.

- More generally, for vector spaces $U_1, \ldots, U_n$ over $\mathbb{K}$, any permutation $\sigma \in \mathscr{S}_n$ determines a *tensor permutation*

$$\pi_\sigma : U_1 \otimes \cdots \otimes U_n \to U_{\sigma(1)} \otimes \cdots \otimes U_{\sigma(n)}$$

by linearisation of the multilinear map

$$(u_1, \ldots, u_n) \mapsto u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(n)}.$$

- For a vector space $U$ over $\mathbb{K}$, the maps $\mathscr{S}_n \to GL(U^{\otimes n})$, $\sigma \mapsto \pi_\sigma$ are group representations of some importance.

# A tensor product/multilinear map isomorphism

### Proposition

*Let* $U_1, \ldots, U_n$ *and* $W$ *be vector spaces over* $\mathbb{K}$*, and let* $\tau$ *denote the map* $U_1 \times \cdots \times U_n \to U_1 \otimes \cdots \otimes U_n$*,* $(u_1, \cdots, u_n) \mapsto u_1 \otimes \cdots \otimes u_n$*. Then the following defines an isomorphism:*

$$\Phi : L(U_1 \otimes \cdots \otimes U_n; W) \to ML(U_1, \ldots, U_n; W), \quad A \mapsto A \circ \tau.$$

### Proof.

The linearisaton map $ML(U_1, \ldots, U_n; W) \to L(U_1 \otimes \cdots \otimes U_n; W)$ is easily seen to be inverse to $\Phi$, so the result follows by the manifest linearity of $\Phi$. □

### Corollary

*For vector spaces $U$ and $V$ over $\mathbb{K}$ there are natural isomorphisms*

$$(U \otimes V)^{\mathrm{dual}} \cong ML(U, V; \mathbb{K}) \cong L(U; V^{\mathrm{dual}}).$$

### Remarks

- $U \otimes V$ is thus a "pre-dual" of the vector space $ML(U, V; \mathbb{K})$.
- *Cf.*
$$\mathcal{F}_{00}(S \times T; \mathbb{K})^{\mathrm{dual}} \cong \mathcal{F}(S \times T; \mathbb{K}) \cong \mathcal{F}(S; \mathcal{F}(T; \mathbb{K})).$$

### Proposition (Tensor product of algebras)

*Let $\mathcal{A}$ and $\mathcal{B}$ be algebras over $\mathbb{K}$. Then the linear tensor product $\mathcal{A} \otimes \mathcal{B}$ has a natural algebra structure, uniquely determined by the identity*

$$(a \otimes b)(a' \otimes b') = aa' \otimes bb' \quad (a, a' \in \mathcal{A},\ b, b' \in \mathcal{B}). \qquad (*)$$

*If $\mathcal{A}$ and $\mathcal{B}$ are unital then so is $\mathcal{A} \otimes \mathcal{B}$; if $\mathcal{A}$ and $\mathcal{B}$ are involutive algebras then there is a unique algebra involution on $\mathcal{A} \otimes \mathcal{B}$ satisfying*

$$(a \otimes b)^* = a^* \otimes b^* \quad (a \in \mathcal{A},\ b \in \mathcal{B}).$$

### Remark

This is consistent with the algebra structure induced on the linear tensor product $L(U) \otimes L(V)$, for vector spaces $U$ and $V$ over $\mathbb{K}$, by virtue of its natural realisation in $L(U \otimes V)$.

Proof.

Since simple tensors linearly span $\mathcal{A} \otimes \mathcal{B}$ and the right-hand side of (*) is manifestly bilinear in $(a, b)$ and in $(a', b')$, there is at most one algebra product on $\mathcal{A} \otimes \mathcal{B}$ satisfying (*).
Let $\mathcal{C} = \mathcal{A} \otimes \mathcal{B}$ and, identifying $\mathcal{C} \otimes \mathcal{C}$ with $\mathcal{A} \otimes \mathcal{B} \otimes \mathcal{A} \otimes \mathcal{B}$ (by associativity), let $M : \mathcal{C} \otimes \mathcal{C} \to \mathcal{C}$ be the linearisation of the multilinear map

$$\mathcal{A} \times \mathcal{B} \times \mathcal{A} \times \mathcal{B} \to \mathcal{C}, \quad (a, b, a', b') \mapsto aa' \otimes bb'.$$

Noting that

$$M(c_1 \otimes M(c_2 \otimes c_3)) = a_1 a_2 a_3 \otimes b_1 b_2 b_3 = M(M(c_1 \otimes c_2) \otimes c_3)$$

when $c_i = a_i \otimes b_i$ for $a_i \in \mathcal{A}$ and $b_i \in \mathcal{B}$ $(i = 1, 2, 3)$, it follows by trilinearity that

$$(c, c') \mapsto cc' := M(c \otimes c') \quad (c, c' \in \mathcal{C})$$

defines an algebra product on $\mathcal{C}$ satisfying (*). □

## LA-10

- Hahn–Banach Extension Theorem

### Theorem (Hahn–Banach)

*Let $(V, \ell)$ be a vector space over $\mathbb{R}$ together with a functional on $V$, which is sublinear: $\forall_{v,v' \in V} \forall_{t \geqslant 0}\ \ell(v + v') \leqslant \ell(v) + \ell(v'),\ \ell(tv) = t\ell(v)$.*
*Let $U$ be a subspace of $V$ and let $\varphi_0 \in U^{\mathrm{dual}}$ be dominated by $\ell$:*

$$\forall_{u \in U}\ \varphi_0(u) \leqslant \ell(u).$$

*Then there is $\varphi \in V^{\mathrm{dual}}$ which extends $\varphi_0$ and is also dominated by $\ell$.*

### Theorem (Hahn–Banach II)

*Let $(V, p)$ be a vector space over $\mathbb{K}$ together with a seminorm on $V$, thus $\forall_{v,v' \in V} \forall_{\lambda \in \mathbb{K}}\ p(v + v') \leqslant p(v) + p(v'),\ p(\lambda v) = |\lambda| p(v))$.*
*Let $U$ be a subspace of $V$ and let $\varphi_0 \in U^{\mathrm{dual}}$ be dominated by $p$:*

$$\forall_{u \in U}\ |\varphi_0(u)| \leqslant p(u).$$

*Then there is $\varphi \in V^{\mathrm{dual}}$ which extends $\varphi_0$ and is also dominated by $p$.*

### Remarks

- Banach limits on $\ell^\infty_{\mathbb{R}}$ give examples: $\ell(x) := \sup_n x_n$, $U = c$, $\varphi_0(u) = \lim u_n$.
- The above results are of vital importance in Functional Analysis.

## **Appendix** (for self-study)

- Nonunital algebras

### Definition

Let $(A_\gamma)_{\gamma\in\Gamma}$ be a family of algebras (respectively *-algebras) over $\mathbb{K}$. Then under componentwise operations:

$$(a_\gamma)\cdot(a'_\gamma) := (a_\gamma a'_\gamma) \quad \big(\text{respectively} \quad (a_\gamma)^* := (a^*_\gamma)\big),$$

$A := \prod_{\gamma\in\Gamma} A_\gamma$ is an algebra (respectively *-algebra) which is unital if each $A_\gamma$ is, with identity $(e_\gamma)_{\gamma\in\Gamma}$, $\sum^{\oplus}_{\gamma\in\Gamma} A_\gamma$ is a subalgebra (respectively *-subalgebra).

### Example

$M_n(A)\oplus M_m(A)$ is naturally realised as a subalgebra of $M_{n+m}(A)$, which is involutive/unital if $A$ is.

### Definition

An ideal $J$ of an algebra $A$ is *essential* if $a\in A$ and $aJ=\{0\}$ implies $a=0$, in other words if the algebra morphism $A\to L(J)$, $a\mapsto L_a|_J$ is injective.

### Examples

- $A$ itself is an essential ideal of $A$ if and only if $A$ is nondegenerate.
- Every nontrivial ideal of $L(V)$ is essential because it contains $L_{00}(V)$ which is.

### Definition

Let $A$ be an algebra over $\mathbb{K}$. Then

$$(a,\lambda)(a'\lambda') := (aa' + \lambda a' + \lambda' a, \lambda\lambda')$$

defines a product on the linear direct sum of $(A, \mathbb{K})$ making it a unital algebra which we denote by $A^+$. If $A$ is involutive then

$$(a,\lambda)^* := (a^*, \lambda^*)$$

defines an algebra involution on $A^+$.
The *conditional unitisation* of $A$ is defined by

$$A^\sim = \begin{cases} A & \text{if } A \text{ is unital,} \\ A^+ & \text{if } A \text{ is nonunital.} \end{cases}$$

### Remark

In $A^\sim$ we usually write $a + \lambda 1$ rather than $(a, \lambda)$ when $A$ is nonunital.

### Examples

- If $A = \mathcal{F}_{00}(S;\mathbb{K})$ then $A^{\sim} \cong \left\{f + \lambda 1 : f \in \mathcal{F}_{00}(S;\mathbb{K}),\ \lambda \in \mathbb{K}\right\} \subset \mathcal{F}_{\mathbb{K}}(S)$.
- If $A = L_{00}(V)$ then $A^{\sim} \cong \left\{T + \lambda I : T \in L_{00}(V),\ \lambda \in \mathbb{K}\right\} \subset L(V)$.
- If $A = c_0$ then $A^{+} \cong c$.
- If $A = C_0(S)$, $S$ locally compact, Hausdorff and noncompact, then $A^{+} \cong C\big(S \cup \{\infty\}\big)$ (see later for one-point compactification).

### Remark

The third example is the special case of the fourth in which $S = \mathbb{N}$, with the discrete topology.

### Proposition

*Let $A$ be an algebra (respectively $*$-algebra) over $\mathbb{K}$.*

(a) *The map $\imath_+ : A \to A^+$, $a \mapsto (a, 0)$, is an algebra (respectively $*$-algebra) monomorphism. Its range is the kernel of the algebra (respectively $*$-algebra) epimorphism $A^+ \to \mathbb{K}$, $(a, \lambda) \mapsto \lambda$, an ideal of $A^+$ of codimension one which moreover is essential if and only if $A$ is nondegenerate.*

(b) *Suppose that $A$ is nonunital. Then $A^+ \to L(A)$, $(a, \lambda) \mapsto L_a + \lambda I$, defines an algebra (respectively $*$-algebra) morphism which is injective if and only if $A$ is nondegenerate.*

(a) *Suppose that $A$ is unital. Then $A^+ \to A \oplus \mathbb{K}$, $(a, \lambda) \mapsto (a + \lambda 1, \lambda)$ defines an algebra (respectively $*$-algebra) isomorphism.*

### Proof.

Exercise. □

### Remarks

- (a) says that the sequence $0 \to A \to A^+ \to \mathbb{K} \to 0$ is exact.
- If $A$ is nondegenerate then $A^\sim$ is naturally realised in $L(A)$.

# Extensions of algebra morphisms

### Proposition

*Let $\varphi : A \to B$ be an algebra (respectively *-algebra) morphism.*

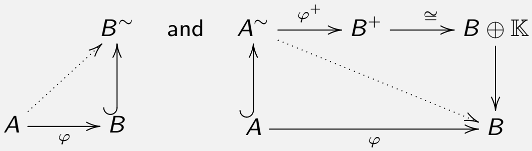
(a) *There is a unique unital algebra (respectively *-algebra) morphism $\varphi^+ : A^+ \to B^+$ which extends $\varphi$ in the sense that $\varphi^+ \circ \imath_+ = \imath_+ \circ \varphi$.*

(b) *There is a unique algebra (respectively *-algebra) morphism $\varphi^\sim : A^\sim \to B^\sim$, which* (i) *extends $\varphi$ in the sense that $\varphi^\sim \circ \imath_\sim = \imath_\sim \circ \varphi$ and* (ii) *is unital if $A$ is nonunital.*
*(Here $\imath_\sim$ denotes $\imath_+$ if $A$ is nonunital and* $\mathrm{id}_A$ *otherwise.)*

*If $\varphi$ is injective (respectively surjective) then $\varphi^\sim$ and $\varphi^+$ are too.*

### Proof.

Exercise. For (b) consider the commutative diagrams




for the cases where $A$ is unital but $B$ is not, respectively $B$ is unital but $A$ is not. $\square$

# Spectrum for elements of a nonunital algebra

### Definition

Let $A$ be a nonunital algebra.
Then, for $a \in A$, the spectrum and spectral radius of $a$ are defined by

$$\sigma(a) := \sigma_{A^+}\big(\imath_+(a)\big) \quad \text{and} \quad r_\sigma(a) := \sup\big\{|\lambda| : \lambda \in \sigma(a)\big\} \subset [0, \infty].$$

### Remarks

- Thus, for any algebra $A$, $\sigma(a) = \sigma_{A^\sim}\big(\imath_\sim(a)\big)$ and $r_\sigma(a) = r_\sigma\big(\imath_\sim(a)\big)$
- If $A$ is nonunital then $0 \in \sigma(a)$.

**Proposition**

*Let $\varphi : A \to B$ be an algebra morphism and let $a \in A$. Then*

$$\sigma\big(\varphi(a)\big) \subset \sigma(a) \cup \{0\}.$$

*If $\varphi^{\sim}$ is unital then $\sigma\big(\varphi(a)\big) \subset \sigma(a)$.*

**Proof.**

The second part is obvious since $xy = 1$ in $A^{\sim}$ implies $\varphi^{\sim}(x)\varphi^{\sim}(y) = \varphi^{\sim}(1)$ in $\varphi^{\sim}(A^{\sim}) \subset B^{\sim}$. Suppose therefore that $\varphi^{\sim}$ is nonunital, and set

$$p = 1 - \varphi^{\sim}(1) \in B^{\sim}.$$

Let $\lambda \in \rho(a) \setminus \{0\}$ and let $c$ be the inverse of $(\lambda 1 - a)$ in $A^{\sim}$. Then $\varphi(A)p = p\varphi(A) = \{0\}$ so

$$\big(\lambda 1 - \varphi(a)\big)\big(\varphi(c) + \lambda^{-1}p\big) = \varphi^{\sim}\big((\lambda 1 - a)c\big) + p = 1$$

and similarly $\big(\varphi(c) + \lambda^{-1}p\big)\big(\lambda 1 - \varphi(a)\big) = 1$. Thus $\lambda \in \rho\big(\varphi(a)\big)$ and the first part follows. □

### Corollary

*Let $C$ be a subalgebra of an algebra $A$. Then*

$$\sigma_A(c) \subset \sigma_C(c) \cup \{0\} \quad (c \in C).$$

*Moreover if $A$ is unital and $1_A \in C$ then $\sigma_A(c) \subset \sigma_C(c)$ for all $c \in C$.*

### Proof.

Apply the proposition to the inclusion map $C \mapsto A$. □

### Proposition

*Let $A$ be a unital algebra. Then*

$$\sigma_{L(A)}(L_a) = \sigma(a) \quad (a \in A).$$

### Proof.

By the proposition LHS $\subset$ RHS. Let $x \in A$ be such that $L_x$ is invertible in $L(A)$. Then, setting $c = (L_x)^{-1}1_A$ and using the right $A$-linearity (exercise) of $(L_x)^{-1}$,

$$xc = L_x c = 1_A \quad \text{and} \quad cx = (L_x)^{-1}(1_A)x = (L_x)^{-1}x = L_x^{-1}L_x 1_A = 1_A$$

so $x$ is invertible. Thus $\rho_{L(A)}(L_a) \subset \rho_A(a)$ for each $a \in A$ and the result follows. □

### Definition

Let $A$ be a complex algebra. A *character* of $A$ is a nonzero algebra morphism $A \to \mathbb{C}$. The collection of all characters of $A$ is denoted by $\Omega_A$ and called the *character space* of $A$ or *spectrum* of $A$.

### Lemma

*Let $A$ be a commutative complex algebra, let $\varphi \in \Omega_A$ and let $a \in A$. Then*

(a) $\varphi^{\sim}$ *is unital;*

(b) $\varphi(a) \in \sigma(a)$*;*

(c) $\left|\varphi(a)\right| \leqslant r_\sigma(a)$.

### Proof.

(a) Since $\varphi$ is nonzero, $\operatorname{Ran}\varphi = \mathbb{C}$. It follows that $\varphi^{\sim}(1) \in \mathbb{C}$ satisfies $\varphi^{\sim}(1)z = z$ for all $z \in \mathbb{C}$, thus $\varphi^{\sim}(1) = 1$.
(b) Since $\varphi^{\sim}(1) = 1$, $\left(\varphi(a)1_{A^{\sim}} - a\right) \in \operatorname{Ker}\varphi^{\sim}$ and so $\left(\varphi(a)1_{A^{\sim}} - a\right)$ cannot be invertible, thus $\varphi(a) \in \sigma(a)$.
(c) follows from (b). □

This simple result is important in the Banach algebra and $C^*$-algebra context.

# Glossary of notation

| | |
|---|---|
| $\subset$ | Subset |
| $\subsetneq$ | Proper subset |
| $\operatorname{card} S$ | Cardinality of a set $S$ |
| $\subset\subset$ | Subset of finite cardinality |
| $\mathcal{P}(S)$ | Power set of $S$, thus $\operatorname{card} \mathcal{P}(S) = 2^{\operatorname{card} S}$ |
| $\mathcal{F}(S; T)$ | The collection of functions $S \to T$, for sets $S$ and $T$ |
| $\mathcal{F}_{00}(S; T)$ | $\{f \in \mathcal{F}(S; T) : \operatorname{supp} f \subset\subset S\}$ |
| $\mathbb{N}$ | $\{1, 2, 3, \ldots\}$ |
| $\aleph_0$ | (aleph-null) $\operatorname{card} \mathbb{N}$ |
| $A_+$ | $A \cap [0, \infty[$ for $A \subset \mathbb{R}$, e.g. $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$ |
| $L(U; V)$ | Vector space of linear maps from vector space $U$ to $V$ |
| $\operatorname{rank} T$ | $\dim(\operatorname{Ran} T)$, for a linear map $T$ |
| $L_{00}(U; V)$ | $\{T \in L(U; V) : \operatorname{rank} T < \infty\}$ |
| $V^{\mathrm{dual}}$ | Dual space of a vector space $V$ over $\mathbb{K}$: $L(V; \mathbb{K})$ |
| $\operatorname{Lin} E$ | Linear span of a subset $E$ of a vector space |
| $\mathrm{F}_{\mathbb{K}}(S)$ | Free group on the set $S$ over the field $\mathbb{K}$. |

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