FUNCTIONAL ANALYSIS via MAGIC

TOPOLOGY (Topic 2)

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This section contains a rapid review of point-set topology, concentrating on aspects most relevant for Functional Analysis. Later we shall be primarily concerned with metric topologies and more specifically norm-topologies. The main exception to this is the the weak*-topology on dual Banach spaces, whose proper understanding requires much of the present material.

Some further topology is contained in the section Metric Spaces.

T-1

- Topological space
- Metric space; metrisability
- Relative topology
- Neighbourhoods
- Neighbourhood base; first-countability
- Closed sets; convergence
- Interior, closure and boundary
- Density and separability

Definition (Topological space)

A *topology* on a set S is a family τ of subsets of S satisfying:

(Ti) $\emptyset, S \in \tau$;

(Tii) $\mathcal{U} \subset \tau \implies \bigcup \mathcal{U} \in \tau$;

(Tiii) $U, V \in \tau \implies U \cap V \in \tau$.

In other words, τ must contain the whole set and empty set, and be closed under arbitrary unions and finite intersections. The elements of τ are called the *open sets* of the topology. The pair (S, τ) is called a *topological space*.

Thus

$$\{\emptyset, S\} \subset \tau \subset \mathcal{P}(S)$$

where $\mathcal{P}(S)$ denotes the *power set* of S, consisting of *all* of the subsets of S.

Examples: Discrete and Trivial

Example (Discrete topologies)

For any set S there is a *discrete topology* on S:

$$\tau = \mathcal{P}(S)$$

in which every subset is open.

Example (Trivial topologies)

For any set S there is the *trivial topology* on S:

$$\tau = \{\emptyset, S\}$$

in which only those sets that *have* to be open are.

Definition (Pseudo-metric space)

A *pseudo-metric* on a set E is a map

$$d : E \times E \to [0, \infty[$$

satisfying:

(Mi) $d(x, z) \leqslant d(x, y) + d(y, z)$;

(Mii) $d(y, x) = d(x, y)$;

(Miii) $d(x, x) = 0$;

for all $x, y, z \in E$. It is a *metric* if, furthermore,

(Miv) $d(x, y) = 0 \implies x = y$.

Consequence

$$\left| d(x, y) - d(x, z) \right| \leqslant d(y, z)$$

for all $x, y, z \in E$.

Open balls in a (pseudo-)metric space

Definition (Open balls)

In a pseudo-metric space E, the set

$$B_r^E(a) := \{x \in E : d(x, a) < r\}$$

is called the *open ball in E with centre a and radius r* $(a \in E,\ r > 0)$.

Lemma (On open balls)

Let E be a pseudo-metric space.

(i) *For all $a \in E$,*

$$E = \bigcup_{r>0} B_r^E(a)$$

(ii) *For all $a, b \in E$ and $r, s > 0$,*

$$x \in B_r^E(a) \cap B_s^E(b) \quad \Longrightarrow \quad B_t^E(x) \subset B_r^E(a) \cap B_s^E(b)$$

where $t = \min\{r - d(a,x), s - d(b,x)\}$.

Proposition (Topology of a pseudo-metric space)

Let (E, d) be a pseudo-metric space. Calling a subset U of E "open" if for each element x of U there is a ball centred at x contained within U, we obtain a topology for E with respect to which each open ball is an open set:

$$\tau := \left\{ U \subset E \middle|\ \forall_{x \in U}\ \exists_{r>0} :\ B_r^E(x) \subset U \right\}$$

Proof.

Exercise. [Hint: The required property is trivially satisfied by $\emptyset$; the lemma does the rest.] □

Terminology

The *topology induced* by d.

Definition (Metrisability)

A topological space (S, τ) is *(pseudo-)metrisable* if τ is induced by a (pseudo-)metric on S.

Example (Discrete metric)

Discrete spaces are metrisable, by the *discrete metric*:

$$d(x, y) := \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}.$$

Remark

In a discrete metric space E:

$$B_r(a) = \begin{cases} \{a\} & \text{for } 0 < r \leqslant 1 \\ E & \text{for } r > 1 \end{cases}.$$

A base for a pseudo-metrisable topology

Proposition (Balls as base for pseudo-metric topologies)

Let τ be the topology induced by a pseudo-metric on a set E. Then

$$\tau = \left\{ \bigcup \mathcal{B}_0 : \mathcal{B}_0 \subset \mathcal{B} \right\}$$

where $\mathcal{B}$ is the collection of open balls:

$$\mathcal{B} := \left\{ B_r(a) : a \in E,\ r > 0 \right\}.$$

Proof.

Exercise. □

Example (Standard topology on $\mathbb{R}^n$)

The topology induced on $\mathbb{R}^n$ by the *Euclidean metric*

$$d(x, y) := \left(\sum_{i=1}^{n} |y_i - x_i|^2 \right)^{1/2}$$

is referred to as the *standard topology* on $\mathbb{R}^n$.

Proposition (Countable unions of open balls)

Every open set of $\mathbb{R}^n$ is expressible as a countable union of open balls.

Proof.

Exercise. [Hint: consider the family

$$\mathcal{B}_0 := \left\{ B_r(q) : q \in \mathbb{Q}^n,\ r \in \mathbb{Q},\ r > 0 \right\}.]$$

□

Proposition (Quotient metric space of a pseudo-metric space)

Let (E, d) be a pseudo-metric space. Then:

(a)
$$x \sim y \quad \text{if} \quad d(x, y) = 0$$
defines an equivalence relation on E;

(b) $\widetilde{d}([x], [y]) := d(x, y)$ *defines a metric on the quotient set $E/\!\sim$;*

(c) *the quotient map $q : E \to E/\!\sim$ satisfies*
$$q^{-1}(B_r^{E/\sim}([a])) = B_r^E(a)$$
for each $a \in E$ and $r > 0$.

Proof.
Exercise. □

Definition (Relative topology)

Given a topological space (S, τ), every subset A of S has an *induced topology* (also called the *relative topology*)

$$\tau_R := \{U \cap A : U \in \tau\}$$

Example (Discrete spaces as relative topologies)

Let A be a subset of $\mathbb{R}$ satisfying

$$\inf \{|y - x| : x, y \in A,\ x \neq y\} > 0.$$

For example, A could be any finite subset of $\mathbb{R}$ or any subset of $\mathbb{Z}$. Then the relative topology on A for the standard topology on $\mathbb{R}$ is the discrete topology on the set A.

Exercise

Check this claim.

Neighbourhood – definition and notation

Definition (Neighbourhoods)

Let (S, τ) be a topological space, let $a \in S$ and let $U \subset S$. Then U is an *open neighbourhood of a* if U is an open set containing a:

$$a \in U \quad \text{and} \quad U \in \tau.$$

H is a *neighbourhood of a* if H contains an open neighbourhood of a:

$$\exists_{U \in \tau} : \quad a \in U \quad \text{and} \quad U \subset H.$$

Notation

$$\mathcal{N}(x) := \{H \subset S : H \text{ is a neighbourhood of } x\}.$$

Definition (Neighbourhood base)

Let $\mathcal{N} \subset \mathcal{N}(x)$ for an element x of a topological space (S, τ). Then $\mathcal{N}$ is called a *neighbourhood base at x* if every neighbourhood of x contains a set from $\mathcal{N}$:

$$\forall_{H \in \mathcal{N}(x)} \exists_{E \in \mathcal{N}} : \quad E \subset H$$

Example (Balls as a neighbourhood base)

Let (S, τ) be a pseudo-metrisable topological space. Then, for any pseudo-metric inducing the topology τ,

$$\left\{ B_{1/n}(a) : n \in \mathbb{N} \right\}$$

is a neighbourhood base at a.

Proposition (Neighbourhood characterisation of openness)

Let (S, τ) be a topological space and let $A \subset S$. Then TFAE:

(i) *A is open;*

(ii) *each element of A has a neighbourhood contained in A*

$$\forall_{x \in A}\ \exists_{H \in \mathcal{N}(x)} : \quad H \subset A;$$

(iii) *each element of A has an open neighbourhood contained in A*

$$\forall_{x \in A}\ \exists_{U \in \mathcal{N}(x) \cap \tau} : \quad U \subset A;$$

(iv) *A is a union of open neighbourhoods of its elements*

$$A = \bigcup \Big\{ U \in \mathcal{N}(x) \cap \tau : x \in A,\ U \subset A \Big\}.$$

Neighbourhood characterisation of openness – proof

Proof.
If (i) holds, then A itself is a neighbourhood of each of its elements, so (ii) holds.

(ii) implies (iii) by the definition of a neighbourhood.

Clearly (iii) implies (iv) and (iv) implies (i) by Axiom (Tii). □

Remark
$\mathcal{N}(x)$ may be replaced by any open neighbourhood base at x in the above proposition.

First-countable topological spaces – definition and example

Definition (First-countability)

A topological space (S, τ) is *first-countable* if each element of S has a countable neighbourhood base.

Example

Pseudo-metrisable spaces are first-countable.

Closed sets – definition

Definition (Closed sets)

Let (S, τ) be a topological space. A subset F of S is *closed* if its complement in S is open:

$$S \setminus F \in \tau.$$

Remark

By de Morgan's laws the collection $\mathcal{C}$ of closed sets of a topological space (S, τ) contains the empty set and the whole set S, and is closed under arbitrary intersection and finite union:

- $S, \emptyset \in \mathcal{C}$
- $\mathcal{F} \subset \mathcal{C} \implies \bigcap \mathcal{F} \in \mathcal{C}$
- $E, F \in \mathcal{C} \implies E \cup F \in \mathcal{C}$

Definition (Convergence of sequences)

Let (S, τ) be a topological space, let (x_n) be a sequence in S and let $a \in S$. Then (x_n) *converges to a* if, for every neighbourhood H of a, (x_n) is *eventually in* H:

$$\forall_{H \in \mathcal{N}(a)} \; \exists_{N \in \mathbb{N}} \; \forall_{n \geqslant N} : \quad x_n \in H$$

Notation: $x_n \to a$.

Remarks

- Since $\mathcal{N}(a)$ may be replaced by any neighbourhood base at a, this agrees with familiar notions of convergence from analysis.
- For topological spaces which are *not* first-countable one has to work with a generalisation of sequences called *nets*.
- If a sequence (x_n) and elements a and b in S satisfy $x_n \to a$ and $x_n \to b$ then we cannot conclude that $a = b$ without further information. For example, consider the case where (S, τ) is pseudo-metrised by d. In that case all we could conclude is that $d(a, b) = 0$. We shall revisit this question of the uniqueness of limits later.

Proposition (Closed sets and limits of sequences)

Let (S, τ) be a topological space.

(a) *Suppose that F is closed. Then, for every element a of S and sequence (x_n) in F, $x_n \to a$ implies $a \in F$.*

(b) *The converse holds when S is first-countable.*

Proof.

(a) Let (x_n) be a sequence in F converging to a and suppose, for a contradiction, that $a \notin F$. Since F is assumed closed, $S \setminus F$ is a neighbourhood of a so (x_n) is eventually in $S \setminus F$. This contradiction proves (a).

(b) Exercise.

□

Interior, closure and boundary points

Definition (Interior, closure and boundary)

Let (S, τ) be a topological space, let $A \subset S$ and let $a \in S$. Then:

- a is an *interior point of* A if a has a neighbourhood contained in A:

$$\exists_{H \in \mathcal{N}(a)} : \quad H \subset A;$$

- a is a *point of closure of* A if every neighbourhood of a meets A:

$$\forall_{H \in \mathcal{N}(a)} : \quad H \cap A \neq \emptyset;$$

- a is a *boundary point of* A if every neighbourhood of a meets both A and $S \setminus A$:

$$\forall_{H \in \mathcal{N}(a)} : \quad H \cap A \neq \emptyset \quad \text{and} \quad H \cap A^{\complement} \neq \emptyset.$$

The following sets are called respectively the *interior of* A, the *closure* of A and the *topological boundary* of A:

$$\operatorname{Int} A := \{a \in S : a \text{ is an interior point of } A\},$$

$$\overline{A} := \{a \in S : a \text{ is a point of closure of } A\},$$

$$\partial A := \{a \in S : a \text{ is a boundary point of } A\}.$$

Proposition (Interior, closure, boundary and complementation)

Let $A \subset S$ for a topological space (S, τ).

(a) $\operatorname{Int} A \subset A \subset \overline{A}$; $\partial A = \overline{A} \setminus \operatorname{Int} A = \partial \overline{A} = \partial(A^{\complement})$.

(b) $\operatorname{Int} A^{\complement} = (\overline{A})^{\complement}$; $\overline{A^{\complement}} = (\operatorname{Int} A)^{\complement}$.

(c) $\operatorname{Int} A = \bigcup\{G \in \tau : G \subset A\}$, *the largest open set contained in A.*

(d) $\overline{A} = \bigcap\{F : F \text{ is closed}, \ F \supset A\}$, *the smallest closed set containing A.*

Proof.

Exercise. □

Theorem (Kuratowski)

Let A be a subset of a topological space S. The maximum number of distinct sets that may be obtained from A by applying the closure and complementation operations is 14.

Exercise

Try and achieve the maximum with a subset A of $\mathbb{R}$ (in its standard topology).

Definition (Density and separability)

Let $A \subset S$ for a topological space S. Then A is *dense in S* if $\overline{A} = S$. The topological space S is *separable* if it has a countable dense subset.

Example

In $\mathbb{R}$,

$$\operatorname{Int}\mathbb{Q} = \operatorname{Int}(\mathbb{R} \setminus \mathbb{Q}) = \emptyset$$

and

$$\overline{\mathbb{Q}} = \partial\mathbb{Q} = \mathbb{R}$$

In particular, since $\mathbb{Q}$ is countable, $\mathbb{R}$ is separable.

We shall see more interesting examples later.

T-2

- Continuity
- Homeomorphism

Definition (Continuity)

Let (S, σ) and (T, τ) be topological spaces and let $f \in \mathcal{F}(S; T)$. Then f is *continuous* if the inverse image of every open set in T is open in S:

$$\forall_{U \in \tau} : \quad f^{-1}(U) \in \sigma$$

Write $C((S, \sigma); (T, \tau))$ or, once the topologies on S and T are understood, $C(S; T)$ for the collection of such continuous functions.

Example (When every function is continuous)

If σ is the discrete topology on S or τ is the trivial topology on T then *every* function $S \to T$ is continuous

$$C((S, \sigma); (T, \tau)) = \mathcal{F}(S; T).$$

Continuity – closed sets characterisation

Proposition (Continuity by closed sets)

Let $f \in \mathcal{F}(S; T)$, for topological spaces S and T. Then TFAE:

(i) *f is continuous;*

(ii) *$f^{-1}(F)$ is closed for each closed subset F of T.*

Proof.

Exercise. $\square$

Definition (Continuity at a point)

Let (S,σ) and (T,τ) be topological spaces, let $a \in S$ and let $f \in \mathcal{F}(S;T)$. Then f *is continuous at* a if, for every neighbourhood H of $f(a)$, $f^{-1}(H)$ is a neighbourhood of a:

$$\forall_{H \in \mathcal{N}\left(f(a)\right)} : \quad f^{-1}(H) \in \mathcal{N}(a).$$

Remark

H may be assumed to be open or from a neighbourhood base at $f(a)$.

Proposition (Continuity $\equiv$ continuity at each point)

Let (S, σ) and (T, τ) be topological spaces and let $f \in \mathcal{F}(S; T)$. Then the following are equivalent:

(i) *f is continuous;*

(ii) *f is continuous at a, for all $a \in S$.*

Proof.

Exercise. □

Proposition (Continuity by limits of sequences)

Let $f \in \mathcal{F}(S; T)$, for topological spaces (S, σ) and (T, τ), and let $a \in S$.

(a) *If f is continuous at a then f is sequentially continuous that is, for every sequence (x_n) in S converging to a, the sequence $(f(x_n))$ converges to $f(a)$.*

(b) *The converse holds if (S, σ) is first-countable.*

Proof.

Exercise. □

Proposition (Continuity under composition)

Let (R, ρ), (S, σ) and (T, τ) be topological spaces, let $f \in \mathcal{F}(R; S)$, let $g \in \mathcal{F}(S; T)$ and let $a \in R$:

(a) *if f is continuous at a and g is continuous at $f(a)$, then $g \circ f$ is continuous at a;*

(b) *if f and g are continuous, then so is $g \circ f$.*

Proof.

Exercise. □

Definition (Homeomorphism)

Let (S, σ) and (T, τ) be topological spaces and let $f \in \mathcal{F}(S; T)$. Then f is a *homeomorphism* if:

- f is continuous;
- f is bijective;
- f^{-1} is continuous.

We say that (S, σ) and (T, τ) are *homeomorphic* when there is such a map f.

Remark

Homeomorphic spaces are *topologically* indistinguishable (e.g. doughnuts and teacups).

Proposition (Homeomorphism group)

The collection of homeomorphisms of a topological space (S, σ) forms a subgroup of the group of bijections from S to itself.
Notation: $\mathrm{Homeo}(S, \sigma)$.

Proof.

Exercise. □

Proposition (All open intervals are homeomorphic)

$\mathbb{R}$ *is homeomorphic to each of its open subintervals in their relative topology.*

Proof.

Exercise. [Hint: the functions

$$\begin{aligned} f &:]-1,1[\to \mathbb{R}, \quad t \mapsto \tan(\pi t/2) \\ g &:]0,1[\to]a,\infty[, \quad t \mapsto t/(1-t)+a \\ h &:]0,1[\to]a,b[, \quad t \mapsto (1-t)a+tb \end{aligned}$$

are all homeomorphisms.] □

Example (Nonhomeomorphic continuous bijection)

Let $T =]0,1[$ and $S = \bigcup_{n=1}^{\infty}(I_n + n)$ where $I_n = [\frac{1}{n+1}, \frac{1}{n}[$. Thus, $T = \bigcup_{n=1}^{\infty} I_n$. The function $g : T \to S$ determined by

$$g|_{I_n}(t) = t + n \quad (n \in \mathbb{N})$$

is bijective and discontinuous at the points $\{\frac{1}{n+1} : n \in \mathbb{N}\}$; however, the inverse function is perfectly continuous. (The topologies on S and T are the relative topologies inherited from $\mathbb{R}$.)

Remark

In functional analysis examples of this phenomena are easy to come by.

T-3

- Hausdorff property

Definition (Hausdorff spaces)

A topological space (S, τ) is *Hausdorff* if distinct elements of S have disjoint neighbourhoods:

$$\forall_{a \neq b \in S} \ \exists_{H \in \mathcal{N}(a),\ E \in \mathcal{N}(b)} : \quad H \cap E = \emptyset$$

equivalently

$$\forall_{a \neq b \in S} \ \exists_{H \in \mathcal{N}(a)} : \quad S \setminus H \in \mathcal{N}(b)$$

Exercise. Another equivalent is

$$\forall_{a \neq b \in S} \exists_{H \in \mathcal{N}(a) \cap \tau,\, E \in \mathcal{N}(b) \cap \tau} : \quad H \cap E = \emptyset.$$

Example (Metrisable $\equiv$ Hausdorff + pseudo-metrisable)

A pseudo-metrisable topological space is Hausdorff if and only if it is metrisable.

Proposition (Closure of singleton sets in Hausdorff spaces)

Let (S, τ) be a Hausdorff topological space and let $a \in S$. Then $\{a\}$ is closed.

Proof.

For each $x \in S \setminus \{a\}$ there is $U \in \mathcal{N}(x) \cap \tau$ such that $a \notin U$, so $U \subset S \setminus \{a\}$. Therefore $S \setminus \{a\}$ is open (by the Neighbourhood Characterisation of Openness) and so $\{a\} = S \setminus (S \setminus \{a\})$ is closed. □

Proposition (Uniqueness of limits in a Hausdorff space)

Let (x_n) be a sequence in a Hausdorff topological space. If $x_n \to a$ and $x_n \to b$, then $a = b$.

Proof.

Exercise. □

Remark

The same holds for nets.

T-4

- Generating topologies; bases and subbases; second-countability
- Initial topologies

Remark

If a set S has at least two elements then it has a variety of topologies.

Proposition

The collection of all possible topologies on a given set S is partially ordered by inclusion.

Proof.

Exercise. □

Definition (Finer/coarser topologies)

When $\tau_1 \leqslant \tau_2$ we say that the topology τ_2 is *finer* than the topology τ_1. Thus, the discrete topology on S is the finest topology on S and the trivial topology is the coarsest.

Proposition (Generated topologies)

Let S be a set.

(a) *Let Σ be a family of topologies on S. Then $\bigcap \Sigma$ is a topology on S. It is the finest topology on S which is contained in σ for each $\sigma \in \Sigma$.*

(b) *Let $\mathscr{S} \subset \mathcal{P}(S)$. Then there is a coarsest topology on S containing $\mathscr{S}$, the topology generated by $\mathscr{S}$, written $\tau(\mathscr{S})$.*

Proof.

(a) This is a straight-forward verification.

(b) Let $\tau = \bigcap \Sigma$, where Σ is the collection of topologies on S containing $\mathscr{S}$ (such as the discrete topology). By (a) τ is a topology on S; it is clearly the coarsest topology containing $\mathscr{S}$.

□

Lemma (On generating topologies)

Let T be a set.

(a) *Let $\mathscr{S} \subset \mathcal{P}(T)$. Then $\tau(\mathscr{S}) = \tau(\mathcal{B}(\mathscr{S}))$, where $\mathcal{B}(\mathscr{S})$ is the collection of finite (nonempty) intersections of sets from $\mathscr{S}$:*

$$\mathcal{B}(\mathscr{S}) := \left\{ \bigcap \mathscr{S}_0 : \quad \emptyset \underset{\neq}{\subset} \mathscr{S}_0 \subset\subset \mathscr{S} \right\}$$

(b) *Let $\mathcal{B} \subset \mathcal{P}(T)$. Then, setting $\tau_0 = \left\{ \bigcup \mathcal{U} : \mathcal{U} \subset \mathcal{B} \right\}$, TFAE:*

(i) *τ_0 is a topology on T;*

(ii) *$\bigcup \mathcal{B} = T$ and*

$$\forall_{B_1, B_2 \in \mathcal{B}} : \quad B_1 \cap B_2 = \bigcup \left\{ B \in \mathcal{B} : \quad B \subset B_1 \cap B_2 \right\};$$

(iii) *$\tau(\mathcal{B}) = \tau_0$.*

Convention

An empty union gives $\emptyset$.

Proof.

(a) Since $\mathscr{S} \subset \mathcal{B}(\mathscr{S}) \subset \tau(\mathscr{S})$, (a) is obvious.

(b) Since τ_0 is closed under arbitrary unions, it is a topology if and only if it is closed under finite intersections. In view of the identity

$$\bigcup \mathcal{U} \cap \bigcup \mathcal{U}' = \bigcup \{ U \cap U' : \quad U \in \mathcal{U}, \; U' \in \mathcal{U}' \}$$

this holds if and only if (ii) holds. Thus, (i) and (ii) are equivalent. Since

$$\mathcal{B} \subset \tau_0 \subset \tau(\mathcal{B})$$

(i) and (iii) are equivalent too.

□

With the further convention that an empty intersection of subsets of S is understood to be S, we have the following.

Corollary (Characterisation of generated topology)

Let $\mathscr{S} \subset \mathcal{P}(S)$ for a set S. The topology on S generated by $\mathscr{S}$ is the collection of subsets expressible as "a union of finite intersections of sets from $\mathscr{S}$".

Definition (Base and subbase)

Let (S, τ) be a topological space.

(a) Let $\mathcal{B} \subset \tau$. Then $\mathcal{B}$ is a *base* for τ if every open set is a union of sets from $\mathcal{B}$:
$$\tau = \left\{ \bigcup \mathcal{U} : \mathcal{U} \subset \mathcal{B} \right\}$$

(b) Let $\mathscr{S} \subset \tau$. Then $\mathscr{S}$ is a *subbase* for the topology τ if $\mathcal{B}(\mathscr{S})$ is a base or, equivalently, if $\tau(\mathscr{S}) = \tau$.

Example (Base of balls)

The open balls of a pseudo-metric space form a base for the induced topology.

Example (Subbase for $\mathbb{R}$)

A subbase for $\mathbb{R}$ is

$$\Big\{]-\infty, a[\,:\, a \in \mathbb{R} \Big\} \cup \Big\{]a, \infty[\,:\, a \in \mathbb{R} \Big\}$$

Example (Base for discrete topologies)

$\mathcal{B}$ is a base for a discrete topological space S if and only if

$$\mathcal{B} \supset \Big\{ \{a\} : a \in S \Big\}$$

The following is a useful application of subbases.

Proposition (Subbase characterisation of continuity)

Let (S,σ) and (T,τ) be topological spaces, let $\mathscr{S}$ be a subbase for τ and let $f \in \mathcal{F}(S;T)$.

Then TFAE:

(i) *f is continuous;*

(ii) $\forall_{G\in\mathscr{S}}: \quad f^{-1}(G) \in \sigma$.

Proof.

This is an immediate consequence of the good behaviour of unions and intersections under inverse images: for any $\mathcal{U} \subset \mathcal{P}(\tau)$

$$f^{-1}\left(\bigcup\mathcal{U}\right) = \bigcup\left\{f^{-1}(U) : U \in \mathcal{U}\right\};$$

$$f^{-1}\left(\bigcap\mathcal{U}\right) = \bigcap\left\{f^{-1}(U) : U \in \mathcal{U}\right\}.$$

□

Definition (Second-countability)

A topological space is *second-countable* if it has a countable base.

Remarks

(i) Having a countable *subbase* suffices.

(ii) Second-countable spaces are necessarily first-countable and separable.

Proposition (Pseudo-metrisable + separable $\implies$ second-countable)

A pseudo-metrisable space is second-countable if and only if it is separable.

Proof.

Exercise. □

Proposition (Initial topologies)

Let S be a set and let $\mathcal{F}$ be a family of functions into topological spaces with common source (= domain) S.

(a) *There is a coarsest topology on S with respect to which each function in $\mathcal{F}$ is continuous: the initial topology of $\mathcal{F}$.*

(b) *Let $g \in \mathcal{F}(R; S)$ for a topological space (R, τ). Then g is continuous (with respect to the initial topology of $\mathcal{F}$ on S) if and only if $f \circ g$ is continuous for all $f \in \mathcal{F}$.*

Proof.

(a) Setting

$$\mathscr{S} = \{ f^{-1}(U) : f \in \mathcal{F} \text{ and } U \in \tau_f \},$$

where τ_f is the topology of the target space of f, the topology $\tau(\mathscr{S})$ is easily seen to fit the bill.

(b) Since the above collection $\mathscr{S}$ is a subbase for the initial topology of $\mathcal{F}$ and

$$\forall_{A\in\mathscr{S}} : \quad g^{-1}(A) \in \tau \quad \iff \quad \forall_{f\in\mathcal{F}} \, \forall_{U\in\tau_f} : \quad f^{-1}(g^{-1}(U)) \in \tau,$$

(b) follows for the subbase characterisation of continuity.
(Note $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$.)

□

Remark

From the above proof it is clear that if $\mathscr{S}_f$ is a subbase for the topology τ_f, for each $f \in \mathcal{F}$, then

$$\{ f^{-1}(G) : f \in \mathcal{F},\ G \in \mathscr{S}_f \}$$

is a subbase for the initial topology of $\mathcal{F}$ on S.

When an initial topology is Hausdorff

Proposition (Hausdorff initial topology)

Let S be a set with the initial topology of a family of functions $(f : S \to T_f)_{f \in \mathcal{F}}$ into Hausdorff topological spaces. If $\mathcal{F}$ separates the points of S, that is

$$\forall_{s_1 \neq s_2 \in S} \; \exists_{f \in \mathcal{F}} : \quad f(s_1) \neq f(s_2)$$

then S is Hausdorff.

Proof.

Exercise. □

T-5

- Product topology

Definition

The *product topology* on $T = \prod_{\lambda \in \Lambda} T_\lambda$, for a family of topological spaces $(T_\lambda)_{\lambda \in \Lambda}$, is the initial topology for the family of coordinate maps $(\pi_\lambda)_{\lambda \in \Lambda}$, which are defined by

$$\pi_\lambda\big((x_\alpha)_{\alpha \in \Lambda}\big) = x_\lambda \qquad (\lambda \in \Lambda).$$

Remarks

(i) A function g from a topological space R into a product space $T = \prod_{\lambda \in \Lambda} T_\lambda$ is continuous if and only if each of its "component maps" $g_\lambda := \pi_\lambda \circ g$ is continuous $R \to T_\lambda$ $(\lambda \in \Lambda)$.

(ii) If $\mathscr{S}_\lambda$ is a subbase for the topology of T_λ, for each $\lambda \in \Lambda$, then sets of the form $\prod_{\lambda \in \Lambda} S_\lambda$ where, for some $\mu \in \Lambda$,

$$S_\mu \in \mathscr{S}_\mu \quad \text{and} \quad S_\lambda = T_\lambda \quad \text{for } \lambda \neq \mu$$

form a subbase for the product topology.

(iii) If $\mathcal{B}_\lambda$ is a base for the topology of T_λ, for each $\lambda \in \Lambda$, then sets of the form $\prod_{\lambda \in \Lambda} S_\lambda$ where, for some $\Lambda_0 \subset\subset \Lambda$,

$$S_\lambda \in \mathcal{B}_\lambda \quad \text{for } \lambda \in \Lambda_0 \quad \text{and} \quad S_\lambda = T_\lambda \quad \text{for } \lambda \in \Lambda \setminus \Lambda_0$$

form a base.

(iv) If each T_λ is Hausdorff then so is T.

(v) A sequence $(x^{(k)})_{k \geqslant 1}$ converges to a in T if and only if $x_\lambda^{(k)} \to a_\lambda$ in T_λ for all $\lambda \in \Lambda$.

Example ($\mathbb{R}^n$ again)

The collection of rectangles

$$\{ I_1 \times \cdots \times I_n : I_k \text{ is a bounded open interval for } k = 1, \ldots, n \}$$

forms a base for the product topology of the standard topology on $\mathbb{R}$.

Exercise

Show that the product topology on $\mathbb{R}^n$ coincides with the standard topology on $\mathbb{R}^n$ induced by the Euclidean metric.

Proposition (Countable product of separable spaces)

Let T be a countable product of separable topological spaces. Then T is separable (in the product topology).

Proof.

Exercise. [Hint: If S_n is a countable dense subset of T_n, where $T = \prod_{n \geqslant 1} T_n$, set

$$S = \bigcup_{n \geqslant 1} S_1 \times \cdots \times S_n \times \{t^*_{n+1}\} \times \{t^*_{n+2}\} \times \cdots$$

where $(t^*_n)_{n \geqslant 1}$ is a fixed element of T.] □

Proposition (Countable product of metrisable spaces)

Let $T = \prod_{j \in J} T_j$ be a countable product of (pseudo-)metrisable topological spaces. Then T is (pseudo-)metrisable.

Proof.

For each $j \in J$ let d_j be a pseudo-metric inducing the topology of T_j, with d_j bounded by one if J is infinite. Define a map

$$d : T \times T \to [0, \infty[, \quad (x, y) \mapsto \sum_{j \in J} \alpha_j d_j(x_j, y_j)$$

where $(\alpha_j)_{j \in J}$ in $]0, \infty[$ is chosen so that $\sum_{j \in J} \alpha_j < \infty$. It is easily verified that d is a pseudo-metric which is a metric if each d_j is. It is a nice exercise to verify that d (pseudo-)metrises the product topology. □

Corollary (Continuity of pseudo-metrics)

Let (E, d) be a pseudo-metric space. Then d is continuous $E \times E \to \mathbb{R}$ (with respect to the topology on E induced by the pseudo-metric).

Proof.

By the proposition, the topology on $E \times E$ is pseudo-metrised by the pseudo-metric

$$\widetilde{d} : (s, t) \mapsto d(s_1, t_1) + d(s_2, t_2)$$

In view of the inequality

$$|d(s_1, s_2) - d(t_1, t_2)| \leqslant d(s_1, t_1) + d(s_2, t_2) = \widetilde{d}(s, t) \quad (s, t \in E \times E)$$

d is Lipschitz continuous $(E \times E, \widetilde{d}) \to \mathbb{R}$. Thus, d is continuous with respect to the product topology. □

Lipschitz continuity is discussed in Section: Metric Spaces.

T-6

- Further implications of the Hausdorff property
- Identity Theorem

Proposition (Hausdorff space valued continuous functions have closed graphs)

Let $f \in C(S;T)$ where S and T are topological spaces and T is Hausdorff. Then

$$\mathrm{Graph}(f) := \{(s, f(s)) : s \in S\}$$

is a closed subset of $S \times T$.

Proof.

Let $(a,b) \in S \times T \setminus G$ where $G = \mathrm{Graph}(f)$. Then $b \neq f(a)$ so, since T is Hausdorff, there are disjoint open neighbourhoods U' and V of $f(a)$ and b, respectively. Set $U = f^{-1}(U')$. Then U is open, $a \in U$ and

$$f(U) \cap V \subset U' \cap V = \emptyset$$

so $U \times V$ is an open neighbourhood of (a,b) disjoint from G. Thus $S \times T \setminus G$ is open and so G is closed. □

Proposition (Closedness of diagonals)

Let S be a topological space. Then the diagonal set

$$D := \{(s, s) : s \in S\}$$

is closed in the product space $S \times S$ if and only if S is Hausdorff.

Proof.

Apply the previous proposition with $T = S$ and $f = \mathrm{id}_S$ for the 'if'. Conversely, suppose that D is closed and let $s \neq s' \in S$. Then (s, s') lies in the open set $S \times S \setminus D$, so there is a basic neighbourhood $U \times U'$ of (s, s') contained in $S \times S \setminus D$. But this is equivalent to saying that U and U' are disjoint neighbourhoods of s and s' respectively. It follows that S is Hausdorff. □

Exercise

Generalise this to higher-order products $S \times \cdots \times S$.

Theorem (Identity Theorem for continuous functions)

Let $f_1, f_2 \in C(S; T)$, where S and T are topological spaces and T is Hausdorff:

(a) $E := \{s \in S : f_1(s) = f_2(s)\}$ *is closed;*

(b) *if E is dense in S, then $f_1 = f_2$.*

Proof.

Exercise. [Hint: Note that $E = F^{-1}\big(G \cap (S \times D)\big)$ where $G = \mathrm{Graph}(f)$, $D = \{(t, t) : t \in T\}$ and F is the function

$$S \to S \times T \times T, \quad s \mapsto \big(s, f_1(s), f_2(s)\big) = \big(s, f(s)\big).]$$

□

T-7

- Compactness
- Tychonoff Theorem
- Heine–Borel Theorem

Definition (Compactness)

Let (S, τ) be a topological space and let $K \subset S$. Then K is *compact* if every open cover of K has a finite subcover:

$$\mathcal{U} \subset \tau, \quad K \subset \bigcup \mathcal{U} \quad \implies \quad \exists_{\mathcal{U}_0 \subset\subset \mathcal{U}} : \quad K \subset \bigcup \mathcal{U}_0$$

or, in terms of indexed sets,

$$\{G_\lambda : \lambda \in \Lambda\} \subset \tau, \quad K \subset \bigcup_{\lambda \in \Lambda} G_\lambda$$

$$\implies \quad \exists_{n \in \mathbb{N},\ \lambda_1, \ldots, \lambda_n \in \Lambda} : \quad K \subset G_{\lambda_1} \cup \cdots \cup G_{\lambda_n}$$

or, more verbosely, "if K is contained in the union of a collection of open sets then K is contained in the union of a finite subcollection of those open sets".

Example (Finite sets are compact)

Any finite subset of a topological space is compact.

Remark

Compactness may fruitfully be viewed as a kind of "generalised finiteness" property or "approximate finiteness".

Example (Compactness of the unit interval)

Let I be the unit interval $[0, 1]$. Then I is a compact subset of $\mathbb{R}$ (with its standard topology).

Proof.

Exercise. [Hint: let $\mathcal{U}$ be an open cover of I. Then, for each $t \in I$, $\mathcal{U}$ is an open cover of $I_t = [0, t]$. Define $J = \{t \in I : I_t \text{ has a finite subcover}\}$ and use the fact that the collection of open intervals forms a base for the topology of $\mathbb{R}$.] □

Compactness and closedness related; relative compactness defined

Proposition (Compactness and closedness)

Let A and K be subsets of a topological space (S, τ).

(a) *Suppose that A is closed, K is compact and $A \subset K$. Then A is compact.*

(b) *Suppose that (S, τ) is Hausdorff and A is compact. Then A is closed.*

Definition (Relative compactness)

A subset A of a topological space (S, τ) is *relatively compact* if it is contained in a compact set.

Remark

In a Hausdorff space, relative compactness of A is equivalent to compactness of $\overline{A}$.

Proof.

(a) Let $\mathcal{U}$ be an open cover of A. Then $\mathcal{U} \cup \{A^{\complement}\}$ is an open cover of the compact set K. Therefore, there is $\mathcal{U}_0 \subset\subset \mathcal{U}$ such that $\mathcal{U}_0 \cup \{A^{\complement}\}$ covers K. But then $\mathcal{U}_0$ covers A. Therefore, A is compact.

(b) Let $a \in E \setminus K$. For each $z \in K$, choose disjoint open neighbourhoods G_z and U_z of z and a, respectively. Then $K \subset \bigcup_{z \in K} G_z$ so, by compactness, there is $n \in \mathbb{N}$ and $z_1, \ldots, z_n \in K$ such that $K \subset G$ where G is the open set $G_{z_1} \cup \cdots \cup G_{z_n}$. Set $U = U_{z_1} \cap \cdots \cap U_{z_n}$. Then U is open and

$$a \in U \subset E \setminus G \subset E \setminus K.$$

Thus, U is a neighbourhood of a contained in $E \setminus K$. It follows that $E \setminus K$ is open (by the Proposition on the neighbourhood characterisation of openness) and so K is closed.

□

A continuous image of a compact set is compact

Theorem (Continuous image of a compact)

Let $f : S \to T$ be continuous, for topological spaces S and T, and let $K \subset S$ be compact. Then $f(K)$ is compact.

Proof.

Let $\mathcal{U}$ be an open cover of $f(K)$. Then $\{f^{-1}(U) : U \in \mathcal{U}\}$ is an open cover of the compact set K. So there is $\mathcal{U}_0 \subset\subset \mathcal{U}$ such that

$$K \subset \bigcup \{f^{-1}(U) : U \in \mathcal{U}_0\}.$$

Applying f (as a set map) to both sides yields $f(K) \subset \bigcup \mathcal{U}_0$. Thus, $f(K)$ is compact. $\square$

Continuous bijections compact-to-Hausdorff are homeomorphisms

Proposition (Compact-to-Hausdorff homeomorphism)

Let $f : S \to T$ be a continuous bijection from a compact space into a Hausdorff space. Then f is a homeomorphism.

Proof.

Exercise. [Hint: recall the Proposition on Continuity by Closed Sets.] $\square$

Theorem (Tychonoff)

Let $K = \prod_{\lambda \in \Lambda} K_\lambda$ *where, for each* $\lambda \in \Lambda$, K_λ *is a compact subset of a topological space* T_λ. *Then* K *is compact in the product topology on* $\prod_{\lambda \in \Lambda} T_\lambda$.

Theorem (Heine–Borel)

Let K be a closed and bounded subset of $\mathbb{R}^n$ or $\mathbb{C}^n$. Then K is compact.

Proof.

Since $\mathbb{C}^n$ is identified topologically with $\mathbb{R}^{2n}$ and the identification respects "boundedness", it suffices to deal with the real case.

Choose a closed and bounded interval J such that $J^n \supset K$. Let $f : \mathbb{R} \to \mathbb{R}$ be the affine function which maps the unit interval I onto J:

$$f(t) = (1-t)a + tb \quad \text{for } J = [a, b]$$

Since I is compact and f is continuous, it follows that $J = f(I)$ is compact. By Tychonoff's theorem, J^n is compact. Being a closed subset of a compact set, K is therefore compact. □

T-8

- Final topologies
- Quotient topology

Final topologies and continuous "maps into"

Proposition (Final topologies)

Let T be a set and let $\mathcal{F}$ be a family of functions from topological spaces into T.

(a) *There is a finest topology on T with respect to which each function in $\mathcal{F}$ is continuous:* the final topology of $\mathcal{F}$.

(b) *Let $g \in \mathcal{F}(T; R)$ for a topological space R. Then g is continuous (with respect to the final topology on T) if and only if $g \circ f$ is continuous for all $f \in \mathcal{F}$.*

Proof.

Exercise. □

Quotient topology – example of final topology

Example (Quotient topology)

Let S be a topological space and let $q : S \to S/\!\sim$ be the quotient map associated with an equivalence relation $\sim$ on S. The *quotient topology* on $S/\!\sim$ is the final topology associated with q.

Properties

- U is open in $S/\!\sim \iff \{s \in S : [s] \in U\}$ is open in S
- $q(A)$ is open in $S/\!\sim \iff A_\sim$ is open in S

where $A_\sim := q^{-1}(q(A)) = \bigcup \{[s] : s \in A\}$ is the $\sim$-*saturation* of A.

- $\{[s]\}$ is closed in $S/\!\sim \iff [s]$ is closed in S
- for any $g \in \mathcal{F}(S/\!\sim; R)$ and topological space R,

$$g \text{ is continuous} \iff s \mapsto g\big([s]\big) \text{ is continuous } S \to R$$