FUNCTIONAL ANALYSIS via MAGIC

BANACH ALGEBRAS (Topic 5)

J. Martin Lindsay

Department of Mathematics and Statistics,

Lancaster University

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Introduction

This section contains a brief introduction to the theory of Banach algebras. After algebraic preliminaries, including the Polynomial Spectral Mapping Theorem, examples and basic properties of Banach algebras are given – C^*-algebras being a particularly important class. Invertibility and the spectrum of elements of a unital Banach algebra are treated next, with C. Neumann's criterion and the Neumann series leading to openness of the group of units and differentiability of the inversion map, compactness of spectra and useful expressions for resolvent functions. The section ends with the Gelfand–Beurling Theorem which establishes nonemptiness of spectra and an important identity known as the Spectral Radius Formula, followed by two applications—the Gelfand-Mazur Theorem and, after a brief discussion of types of elements of a C^*-algebra (normal, unitary, projection and similarity), uniqueness of C^*-norms.

BA-1

- Polynomial Spectral Mapping Theorem for complex unital algebras
- Banach algebras; C^*-algebras; examples
- Quotient algebras by closed ideals

Algebras again

Recall the following identity, for an element a of a unital complex algebra:

$$\sigma(\mu 1 + \nu a) = \{\mu\} + \nu\sigma(a), \quad \mu, \nu \in \mathbb{C}.$$

To this we may add the following: when a is invertible,

$$\sigma(a^{-1}) = \sigma(a)^{-1}$$

(exercise), where the RHS is understood to mean $\{\lambda^{-1} : \lambda \in \sigma(a)\}$, and if A is involutive, then

$$\sigma(a^*) = \sigma(a)^*$$

where the RHS is understood to mean $\{\lambda^* : \lambda \in \sigma(a)\}$.

Definition

Let us call the unital *-subalgebra of $\mathcal{F}(\mathbb{C})$ consisting of polynomial functions, the *polynomial algebra*, and denote it by $\mathcal{P}$. For each element a of a unital algebra A, the map

$$\Phi_a^{\mathcal{P}} : \quad \mathcal{P} \to A; \; p \mapsto p(a)$$

is a morphism of unital algebras,. When A is involutive and $a = a^*$, $\Phi_a^{\mathcal{P}}$ is a morphism of unital *-algebras.

Proposition (Polynomial Spectral Mapping Theorem)

Let $a \in A$, a unital complex algebra. If a has a nonempty spectrum then

$$\sigma\big(p(a)\big) = p\big(\sigma(a)\big) \qquad \text{for any } p \in \mathcal{P}.$$

Proof.

If p is a constant, say λ, then (since $\sigma(a)$ is assumed to be nonempty)

$$\sigma\big(p(a)\big) = \sigma(\lambda 1) = \{\lambda\} = p\big(\sigma(a)\big).$$

Assume therefore, without loss of generality, that p is a nonconstant polynomial, which we may further assume is monic, and let $\mu \in \mathbb{C}$. Letting $\prod_{i=1}^{N}(z-\mu_i)^{n_i}$ be the factorisation of $p(z)-\mu$ into its linear factors,

$$\prod_{i=1}^{N}(a-\mu_i 1)^{n_i} = p(a) - \mu 1,$$

moreover $\big\{\mu_i : i = 1, \dots, N\big\} = p^{-1}\big(\{\mu\}\big)$. The result now follows since

$$\begin{aligned} \mu \in \sigma\big(p(a)\big) &\iff \mu_i \in \sigma(a) \text{ for some } i \\ &\iff p^{-1}\big(\{\mu\}\big) \cap \sigma(a) \neq \emptyset \iff \mu \in p\big(\sigma(a)\big). \end{aligned}$$

Note that commutativity of $\{(a-\mu_i 1) : i = 1, \dots, N\}$ is used. □

Normed algebras; C^*-algebras

Definition

A *normed algebra* is an algebra A which is also a NLS such that the norm is submultiplicative:

$$\|aa'\| \leqslant \|a\| \, \|a'\| \quad \text{for all } a, a' \in A.$$

If the algebra is unital then it is usually assumed further that

$$\|1\| = 1.$$

An *involutive normed algebra*, or *normed *-algebra*, is a normed algebra which is also an involutive algebra such that the involution is isometric:

$$\|a^*\| = \|a\| \quad \text{for all } a \in A.$$

A *Banach algebra* is a normed algebra which is complete with respect to the metric induced by its norm. Finally, an involutive Banach algebra whose norm satisfies

$$\|a^* a\| = \|a\|^2 \quad (C^*\text{-relation})$$

is called a C^*-*algebra*.

Basic remarks on Banach algebras

Remarks

- If $\mathcal{C}$ is a closed subalgebra of a Banach algebra A then $\mathcal{C}$ is a Banach-algebra itself.
- If $\mathcal{C}$ is a closed *-subalgebra of an involutive Banach algebra (or C^*-algebra) then $\mathcal{C}$ is an involutive Banach algebra (respectively a C^*-algebra).
- A Banach subalgebra of a Banach algebra A may be unital, without A being unital – or may be unital with a different identity [exercise: find examples of this].
- For a Banach *-algebra to be a C^*-algebra if suffices that

$$\|a^* a\| \geqslant \|a\|^2 \qquad \text{for all } a \in A.$$

- The NLS completion of a

  normed algebra/involutive algebra/'pre-C^*-algebra'

  is respectively a Banach algebra/involutive algebra/C^*-algebra with respect to uniquely defined algebra (respectively, *-algebra) operations.

Examples of Banach algebras and C^*-algebras

Examples

- For any set S, the Banach space $\ell^\infty(S)$ is a commutative unital C^*-algebra under pointwise operations.
- For any set S and normed algebra A, $\ell^\infty(S;A)$ is a normed algebra, which is involutive/complete/C^* if A is.
- For any NLS X, $B(X)$ is a unital normed algebra (under composition); it is a Banach algebra if X is a Banach space, and a C^*-algebra if X is a Hilbert space.
- For any infinite dimensional NLS X, $B_{00}(X) := B(X) \cap L_{00}(X)$ and $B_0(X) := \overline{B_{00}(X)}$ are nonunital subalgebras of $B(X)$, as is

$$K(X) := \{T \in B(X) : T(B_1^X[0]) \text{ is relatively compact}\}$$

- For any Hilbert space h, $B(\mathsf{h})$ is a unital C^*-algebra.
- $L^1(\mathbb{R})$ is a nonunital Banach algebra under the convolution product, given by

$$(f * g)(s) = \int f(s-t)g(t)\,\mathrm{d}t.$$

More basic remarks

- For a subset $\mathscr{S}$ of a C^*-algebra A we denote by $C^*(\mathscr{S})$ the C^*-subalgebra of A *generated by* S.
- For a unital C^*-algebra A and selfadjoint element a,

$$C^*\big(\{1, a\}\big) = \overline{\Phi(\mathcal{P})}$$

where $\Phi = \Phi_a^{\mathcal{P}}$ (exercise).

Quotient Banach algebra

Proposition

*Let J be a closed ideal of a Banach algebra A. Then the quotient norm on the quotient algebra is submultiplicative and so endows A/J with normed algebra structure. If A is an involutive Banach algebra (respectively a C^*-algebra) and J is a *-ideal then A/J is an involutive algebra (respectively C^*-algebra).*

Proof.

Exercise. □

Remark

Closed ideals of a C^*-algebra are automatically *-ideals.

BA-2

- C. Neumann criterion for invertibility; Neumann series
- Openness of $GL(A)$; differentiability of inversion
- Spectral radius; resolvent set and function
- Compactness of spectra; formulae for derivatives of resolvents
- Gelfand–Beurling Theorem
- Gelfand-Mazur Theorem
- Uniqueness of C^*-norms

Invertibility – the C. Neumann series

Theorem

Let A be a unital Banach algebra and let $a \in A$ satisfy $\|a\| < 1$. Then:

(a) $1 - a \in GL(A)$;

(b) $(1-a)^{-1} = \sum_{k=0}^{\infty} a^k$ *(C. Neumann series).*

Proof.

Since $\|a^k\| \leqslant \|a\|^k$ and $\|a\| < 1$, the Neumann series is absolutely convergent and so converges (since A is complete) to s say, and

$$(1-a)s = \lim_{N\to\infty}(1-a)\sum_{k=0}^{N} a^k = \lim_{N\to\infty}(1-a^{N+1}) = 1.$$

Similarly $s(1-a) = 1$. The result follows. □

Remarks

- This is the fundamental criterion for invertibility in a unital Banach algebra.
- The hypothesis $\|a\| < 1$ may clearly be weakened to convergence of the Neumann series (which implies that $a^n \to 0$).

Openness of $GL(A)$; differentiability of inversion

The following consequences of the Neumann series hint at the proposition below: for $\|a\| < 1$,

$$\left\|(1-a)^{-1}\right\| \leqslant (1-\|a\|)^{-1}, \quad \left\|(1-a)^{-1} - 1\right\| \leqslant \|a\|(1-\|a\|)^{-1} \quad \text{and}$$
$$\left\|(1-a)^{-1} - (1+a)\right\| \leqslant \|a\|^2(1-\|a\|)^{-1}.$$

Proposition

Let g be the inversion map on $GL(A)$, $a \mapsto a^{-1}$, for a complex unital Banach algebra A. Then:

(a) *$GL(A)$ is open;*

(b) *g is a homeomorphism of $GL(A)$;*

(c) *g is differentiable, as a map $GL(A) \subset A \to A$, with derivative given by*

$$g'(c)x = -c^{-1}xc^{-1}, \quad \text{for } c \in GL(A),\ x \in A.$$

Proof.

Exercise. [HINT: Use the Neumann criterion, and the fact that $GL(A)$ is a group, to show if $c \in GL(A)$ then $c^{-1}x \in GL(A)$, so $x \in GL(A)$, whenever $\|c - x\| < \|c^{-1}\|^{-1}$. For (b) and (c) show differentiability of g at 1 first.] □

Spectral radius and resolvent (function)

Definition

Let $a \in A$ for a complex unital algebra A. Then

$$\rho(a) := \mathbb{C} \setminus \sigma(a) \quad \text{and} \quad r_\sigma(a) := \sup\left\{|\lambda| : \lambda \in \sigma(a)\right\}$$

are called the *resolvent set* of a and the *spectral radius* of a respectively. The function

$$R_a : \rho(a) \to A, \quad \lambda \mapsto (\lambda 1 - a)^{-1}$$

is called the *resolvent function*.

The *resolvent identity*

$$R_a(\lambda) - R_a(\mu) = (\mu - \lambda) R_a(\lambda) R_a(\mu) \quad (\lambda, \mu \in \rho(a))$$

is often useful.

$\sigma(a) \subset B^{\mathbb{C}}_{\|a\|}[0]$, compact; resolvent differentiated

Lemma

Let $a \in A$ for a complex unital Banach algebra A. Then:

(a) *$\sigma(a)$ is compact and contained in the ball $B^{\mathbb{C}}_{\|a\|}[0]$;*

(b) *for $|\lambda| > \|a\|$,*

$$R_a(\lambda) = \sum_{k=0}^{\infty} \lambda^{-(k+1)} a^k \quad \text{and} \quad \big\|R_a(\lambda)\big\| \leqslant \big(|\lambda| - \|a\|\big)^{-1};$$

(c) *$R_a : \rho(a) \subset \mathbb{C} \to A$ is differentiable and*

$$(R_a)'(\lambda) = -R_a(\lambda)^2 \quad \text{for } \lambda \in \rho(a).$$

Proof.

Exercise. [HINT: Let f_a be the function $\mathbb{C} \to A$, $\lambda \mapsto \lambda 1 - a$. Then $\sigma(a) := f_a^{-1}\big(A \setminus GL(A)\big)$; $|\lambda| > \|a\|$ implies $\|\lambda^{-1} a\| < 1$; and $R_a = g \circ f_a$, where $g : GL(A) \subset A \to A$ is inversion, so the Chain Rule applies.] □

Nonemptiness of spectra; Spectral Radius Formula

Theorem (Gelfand–Taylor and Gelfand-Beurling)

Let $a \in A$ for a complex unital Banach algebra A. Then:

(a) *$\sigma(a)$ is nonempty;*

(b) $r_\sigma(a) = \lim_{n\to\infty} \|a^n\|^{1/n} = \inf \left\{\|a^n\|^{1/n} : n \in \mathbb{N}\right\}$.

Remarks

- (a) has been dubbed the Fundamental Theorem of Banach Algebras.
- (b) is known as the Spectral Radius Formula.
- (b) includes the claim that the limit exists.
- (b) *has to be* important as it relates a quantity extracted from the purely algebraic structure of A with an analytic quantity.

Proof: part (a)

Fix $a \in A$ and let $\varphi \in A^*$. By the Chain Rule and the Lemma,

$$(\varphi \circ R_a)'(\lambda) = -\varphi'\big(R_a(\lambda)\big)R_a(\lambda)^2 = -\varphi\big(R_a(\lambda)^2\big) \quad \text{if } \lambda \in \rho(a).$$

On the other hand, for $|\lambda| > \|a\|$,

$$(\varphi \circ R_a)(\lambda) = \lambda^{-1}\sum_{k=0}^{\infty}\varphi(\lambda^{-k}a^k) \quad \text{and} \quad \big|(\varphi \circ R_a)(\lambda)\big| \leqslant \|\varphi\|\big(|\lambda| - \|a\|\big)^{-1}.$$

(a) In search of a contradiction, assume that $\sigma(a) = \emptyset$. Then $\varphi \circ R_a$ is entire analytic. Being continuous and, by the above, also vanishing at infinity and so bounded, Liouville's Theorem implies that it is constant. Since $\varphi \circ R_a$ vanishes at infinity the constant can only be zero. By the Hahn–Banach Theorem we know that A^* separates the elements of A, so R_a must be identically zero. But this really is nonsense, since each $R_a(\lambda)$ is invertible. We therefore have our contradiction, showing that $\sigma(a) \neq \emptyset$. □

Proof: part (b)

(b) If $a = 0$ then the result is obvious, so assume that $a \neq 0$ and let $\varphi \in A^*$. Then $\varphi \circ R_a : \rho(a) \subset \mathbb{C} \to \mathbb{C}$ is analytic and $\rho(a) \supset \{\lambda \in \mathbb{C} : |\lambda| > r_\sigma(a)\}$ so, by the uniqueness of Laurent expansions, the identity

$$(\varphi \circ R_a)(\lambda) = \sum_{k=0}^{\infty} \lambda^{-(k+1)} \varphi(a^k)$$

is valid for $|\lambda| > r_\sigma(a)$, the series converging absolutely.

Now fix λ such that $|\lambda| > r_\sigma(a)$. The above implies that $\varphi(\lambda^{-k} a^k) \to 0$ as $k \to \infty$. In particular, the family $\{\eta^A(\lambda^{-k} a^k) : k \in \mathbb{N}\}$ is pointwise bounded in A^{**}, and so norm-bounded by the Banach–Steinhaus Theorem. Since the topological bidual embedding $\eta^A : A \to A^{**}$ is isometric, this implies that $\{\lambda^{-k} a^k : k \in \mathbb{N}\}$ is bounded, by M say.
Since $|\lambda| M^{1/k} \to |\lambda|$ as $k \to \infty$, we have

$$\limsup_{k \to \infty} \|a^k\|^{1/k} \leqslant |\lambda|.$$

Therefore $r_\sigma(a) \geqslant \limsup_{k \to \infty} \|a^k\|^{1/k}$.

The Spectral Mapping Theorem implies that $\sigma(a^k) = \sigma(a)^k$ and so if $\lambda \in \sigma(a)$ then $\lambda^k \in \sigma(a^k) \subset B_{\|a^k\|}[0]$, by (a). Thus $r_\sigma(a) \leqslant \|a^k\|^{1/k}$ for each $k \in \mathbb{N}$ and the result follows since we have

$$\liminf_{k\to\infty} \|a^k\|^{1/k} \geqslant \inf\left\{\|a^k\|^{1/k} : k \in \mathbb{N}\right\} \geqslant r_\sigma(a) \geqslant \limsup_{k\to\infty} \|a^k\|^{1/k}.$$

□

Corollary (Gelfand–Mazur)

Let A be a complex unital Banach algebra satisfying

$$GL(A) = A \setminus \{0\}.$$

Then $A = \mathbb{C}1$.

Proof.

Let $a \in A$. Then $\sigma(a) \neq \emptyset$ and, for $\lambda \in \sigma(a)$, $(\lambda 1 - a) \in A \setminus GL(A) = \{0\}$ so $a = \lambda 1$. Thus, $A = \mathbb{C}1$. □

Element types for a C^*-algebra

Definition

Let a be an element of a involutive algebra A. Then a is *normal* if $aa^* = a^*a$, *selfadjoint* (or Hermitian) if $a = a^*$ and (when A is unital) *unitary* if $a^*a = aa^* = 1$; the *real* and *imaginary parts* of a are $\frac{1}{2}(a + a^*)$ and $\frac{1}{2i}(a - a^*)$. If A is a C^*-algebra then we call a a *projection* if $a^2 = a = a^*$ (selfadjoint idempotent), an *isometry* if $a^*a = 1$, a *coisometry* if $aa^* = 1$ and (less commonly) a *similarity* if $a^2 = 1$ and $a = a^*$ (selfadjoint unitary).

Remarks

- If an element a of a unital involutive algebra is invertible, then so is a^* with $(a^*)^{-1} = (a^{-1})^*$.
- For a projection p in a C^*-algebra, the projection $(1 - p)$ is denoted $p^\perp$.
- For an element a of a unital C^*-algebra, a is a projection if and only if $(2a - 1)$ is a similarity.
- The nonunital C^*-algebra $C_0(\mathbb{R})$ has no nontrivial projections.
- For a Hilbert space h, the unital C^*-algebra $B(\mathrm{h})$ has plenty of projections.
- The Spectral Theorem gives a sense in which projections are the building blocks from which all Hilbert space operators may be constructed.

Proposition

Let a be an element of a unital C^*-algebra A. Then

$$\text{(a) } \|a\| = r_\sigma(a^*a)^{1/2}, \text{ and (b) } \|a\| = r_\sigma(a) \text{ if } a \text{ is normal.}$$

Proof.

Since (a) follows from (b) by the C^*-identity, we assume that a is normal. For $k \in \mathbb{N}$, repeated application of the C^*-identity gives

$$\|a\| = \|a^*a\|^{1/2} = \|(a^*a)^2\|^{1/4} = \cdots = \|(a^*a)^{2^k}\|^{2^{-(1+k)}}.$$

Therefore, by normality and another application of the C^*-identity,

$$\|a\| = \|(a^{2^k})^* a^{2^k}\|^{2^{-(1+k)}} = \|a^{2^k}\|^{2^{-k}}$$

The result therefore follows from the Spectral Radius Formula. □

Remark

By the technique of adjoining an identity the unital assumption may be dropped (once the spectrum of elements of nonunital algebras is defined).

Corollary (Uniqueness of C^*-norms)

On a involutive algebra there is at most one complete C^*-norm.