FUNCTIONAL ANALYSIS via MAGIC

NETS AND GENERALISED SERIES (Appendix 2)

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Introduction

In this appendix we give the basic properties of the generalised sequences known as nets. In particular, this allows us to extend the notions of Cauchy and completeness to uniform spaces such as topological vector spaces. It also enables us to discuss generalised series, first in $[0, \infty]$ and then in a Banach space.

NGS-1

- Directed sets and nets
- Completeness $\equiv$ sequential completeness for pseudometrisable uniform spaces

Definition

- A relation $\leqslant$ on a set Γ is a *preorder* if it is transitive ($\alpha \leqslant \beta$, $\beta \leqslant \gamma \implies \alpha \leqslant \gamma$) and reflexive ($\gamma \leqslant \gamma$). Thus a partial order is a preorder which is also antisymmetric ($\alpha \leqslant \beta$, $\beta \leqslant \alpha \implies \alpha = \beta$).
- A *directed set* is a set Γ with a preorder satisfying the 'upward filtering' property:
$$\forall_{\gamma_1,\gamma_2 \in \Gamma}\ \exists_{\gamma \in \Gamma} : \quad \gamma_1 \leqslant \gamma \quad \text{and} \quad \gamma_2 \leqslant \gamma.$$
- A *net* (or *generalised sequence*) in a set S is a family $(s_\gamma)_{\gamma \in \Gamma}$ in S indexed by a directed set.
- A net $(s_\gamma)_{\gamma \in \Gamma}$ in a topological space S *converges* to a point p if
$$\forall_{H \in \mathcal{N}(p)}\ \exists_{\delta \in \Gamma}\ \forall_{\gamma \geqslant \delta} \quad s_\gamma \in H.$$
- A net $(s_\gamma)_{\gamma \in \Gamma}$ in a uniform space (S, υ) is *Cauchy* if
$$\forall_{V \in \upsilon}\ \exists_{\delta \in \Gamma}\ \forall_{\alpha,\beta \geqslant \delta} \quad (s_\alpha, s_\beta) \in V.$$
- A uniform space is *complete* if all of its Cauchy nets converge.

Examples of directed sets and nets

Examples

- For any set Λ, the power set $\mathcal{P}(\Lambda)$ and 'finite power set'
$$\mathcal{P}_{00}(\Lambda) := \{F \in \mathcal{P}(\Lambda) : F \subset\subset \Lambda\}$$
are directed sets under the partial order given by inclusion.
- $\mathbb{N}$ is a directed set under its usual total order; a net indexed by $\mathbb{N}$ is nothing but a sequence.
- If Γ and Λ are directed sets then so is $\Gamma \times \Lambda$ under the *product preorder* given by $(\gamma, \lambda) \leqslant (\gamma', \lambda')$ if $\gamma \leqslant \gamma'$ and $\lambda \leqslant \lambda'$.
- A net $(s_\gamma)_{\gamma\in\Gamma}$ in a uniform space (S, υ) is Cauchy if and only if for every $V \in \upsilon$, $\big((s_\gamma, s_\delta)\big)_{(\gamma,\delta)\in\Gamma\times\Gamma}$ is eventually in V (υ may be replaced by a subbase for υ).
- If (S, υ) is pseudometrised by d then $(s_\gamma)_{\gamma\in\Gamma}$ is Cauchy if and only if $d\big((s_\gamma, s_\delta)\big)_{(\gamma,\delta)\in\Gamma\times\Gamma}$ converges to 0.

Proposition

- *A function $f : S \to T$ between topological spaces is continuous at p if and only if $f(s_\gamma) \to f(p)$ for every net $(s_\gamma)_{\gamma \in \Gamma}$ converging to p.*
- *A topological space S is Hausdorff if and only if any net in S converges to at most one point.*
- *In a uniform space every convergent net is Cauchy.*

Proof.
Exercise. □

Proposition

Let X be a pseudometrisable uniform space. Then X is complete if and only if it is sequentially complete (all of its Cauchy sequences converge).

Proof.

Since every Cauchy sequence is a Cauchy net, completeness implies sequential completeness. Suppose therefore that X is sequentially complete and let d be a pseudometric for X. Now let $(x_\lambda)_{\lambda\in\Lambda}$ be a Cauchy net in X. Fix $\lambda(0) \in \Lambda$ and choose $\lambda(1), \lambda(2), \ldots \in \Lambda$ recursively so that

$$\lambda(n) \geqslant \lambda(n-1) \quad \text{and} \quad d(x_\lambda, x_\mu) < \frac{1}{n} \quad \text{for } \lambda, \mu \geqslant \lambda(n).$$

The sequence $(x_{\lambda(n)})_{n\in\mathbb{N}}$ is then Cauchy and so convergent, to x say; moreover for $\lambda \geqslant \lambda(n)$

$$d(x_\lambda, x) \leqslant d(x_\lambda, x_{\lambda(n)}) + d(x_{\lambda(n)}, x) < \frac{1}{n} + d(x_{\lambda(n)}, x)$$

and the right-hand side converges to 0 as $n \to \infty$. It follows that $x_\lambda \to x$. Thus X is complete, as is our proof □

NGS-2

- Generalised series in $[0, \infty]$
- Fubini Lemma for generalised double series in $[0, \infty]$
- Fubini for absolutely convergent generalised series

$[0,\infty]$

The one-point compactification of $\mathbb{R}_+ = [0,\infty[$, with its usual topology, inherits a total order from that of $[0,\infty[$ in which the point at infinity is greater than all others. This space is therefore written $[0,\infty]$. Arithmetic operations on $[0,\infty[$ are extended as follows

$$a + \infty = \infty + a := \infty$$

$$a \cdot \infty = \infty \cdot a := \begin{cases} \infty & \text{if } a \neq 0 \\ 0 & \text{if } a = 0 \end{cases}$$

$$\frac{1}{\infty} := 0 \quad \text{and} \quad \frac{1}{0} := \infty.$$

Addition remains continuous; multiplication remains continuous except at the points $(0,\infty)$ and $(\infty,0)$; the reciprocal function is now (defined and) continuous everywhere. The function $[0,1] \to [0,\infty]$, $t \mapsto t/(1-t)$ is an order-preserving homeomorphism with inverse $s \mapsto s/(1+s)$.

Lemma

Let $(x_\gamma)_{\gamma\in\Gamma}$ be an increasing net in $[0,\infty]$. Then (x_γ) converges with limit $\sup_{\gamma\in\Gamma} x_\gamma$.

Proof.

Straightforward. □

Lemma on double sums

Lemma

Let $(x_{(a,b)})_{(a,b)\in A\times B}$ *be a doubly indexed family in* $[0,\infty]$ *and let* $F \subset\subset B$*. Then*

(a) $\sup_{(a,b)\in A\times B} x_{a,b} = \sup_{a\in A} \sup_{b\in B} x_{a,b}$

(b) $\sup_{a\in A} \sum_{b\in F} x_{a,b} = \sum_{b\in F} \sup_{a\in A} x_{a,b}$.

Proof.

Straightforward (exercise). □

Definition

For an indexed family $(a_\lambda)_{\lambda\in\Lambda}$ in $[0, \infty]$, the generalised series $\sum_{\lambda\in\Lambda} a_\lambda$ *converges to* x if the corresponding net $(s_F := \sum_{\lambda\in F} a_\lambda)_{F\in\mathcal{P}_{00}(\Lambda)}$ converges to x.

Proposition

Let $(a_\lambda)_{\lambda\in\Lambda}$ *be an indexed family in* $[0, \infty]$. *Then*

(a) $\sum_{\lambda\in\Lambda} a_\lambda$ *converges, with limit* $\sup_{F\subset\subset\Lambda} \sum_{\lambda\in F} a_\lambda$;

(b) *if* $\sum_{\lambda\in\Lambda} a_\lambda < \infty$ *then* $\{\lambda \in \Lambda : a_\lambda \neq 0\}$ *is countable.*

Proof.

Since the net $(s_F := \sum_{\lambda\in F} a_\lambda)_{F\in\mathcal{P}_{00}(\Lambda)}$ is increasing, (a) follows from the lemma and definition.
(b) Suppose that $s := \sum_{\lambda\in\Lambda} a_\lambda < \infty$ and for each $n \in \mathbb{N}$ let $\Lambda_n = \{\lambda \in \Lambda : a_\lambda \geqslant \frac{1}{n}\}$. Then, for any $F \subset\subset \Lambda_n$,

$$\sum_{\lambda\in F} a_\lambda \geqslant \frac{1}{n}\#F$$

so $\#F \leqslant ns$. It follows that $\Lambda_n \subset\subset \Lambda$. The result now follows since

$$\{\lambda \in \Lambda : a_\lambda \neq 0\} = \bigcup_{n\in N} \Lambda_n.$$

$\square$

Proposition (Fubini)

Let $\Lambda = \Gamma \times \Delta$ for sets Γ and Δ, and let $(a_\lambda)_{\lambda \in \Lambda}$ be an indexed family in $[0, \infty]$. Then

$$\sum_{\gamma \in \Gamma} \sum_{\delta \in \Delta} a_{\gamma,\delta} = \sum_{\lambda \in \Lambda} a_\lambda = \sum_{\delta \in \Delta} \sum_{\gamma \in \Gamma} a_{\gamma,\delta}.$$

Proof.

First we note that, for any $F \subset\subset \Lambda$ there are $F_1 \subset\subset \Gamma$ and $F_2 \subset\subset \Delta$ such that $F \subset F_1 \times F_2$. Now the net $(s_F := \sum_{\lambda \in F} a_\lambda)_{F \in \mathcal{P}_{00}(\Lambda)}$ is increasing and so, using the lemma, this implies that

$$\begin{aligned} \sum_{\lambda \in \Lambda} a_\lambda &= \sup_{(F_1, F_2) \in \mathcal{P}_{00}(\Gamma) \times \mathcal{P}_{00}(\Delta)} \sum_{\lambda \in F_1 \times F_2} a_\lambda \\ &= \sup_{F_1 \subset\subset \Gamma} \sup_{F_2 \subset\subset \Delta} \sum_{\gamma \in F_1} \sum_{\delta \in F_2} a_{\gamma,\delta} \\ &= \sup_{F_1 \subset\subset \Gamma} \sum_{\gamma \in F_1} \sup_{F_2 \subset\subset \Delta} \sum_{\delta \in F_2} a_{\gamma,\delta} \\ &= \sum_{\gamma \in \Gamma} \sum_{\delta \in \Delta} a_{\gamma,\delta}, \end{aligned}$$

giving the first equality. The second equality follows by symmetry. □

Definition

Let $(x_\lambda)_{\lambda\in\Lambda}$ be an indexed family in a NLS X. The formal sum $\sum_{\lambda\in\Lambda} x_\lambda$ is called a *generalised series*, and is said to *converge to z* if the corresponding net $(\sum_{\lambda\in F} x_\lambda)_{F\in\mathcal{P}_{00}(\Lambda)}$ converges to z, and to *converge absolutely* if $\sum_{\lambda\in\Lambda} \|x_\lambda\| < \infty$.

Remark

When $\Lambda \in \mathbb{N}$, convergence of $\sum_{i\in\mathbb{N}} x_i$ in the above sense is termed 'unconditional convergence of $\sum_{i=1}^{\infty} x_i$'; it is much stronger than convergence of the series $\sum_{i=1}^{\infty} x_i$ in the usual sense.

Lemma (Characterisation of Convergence of Generalised Series in a Banach Space)

For an indexed family $(x_\lambda)_{\lambda\in\Lambda}$ in a Banach space X, TFAE

(i) *For all $\varepsilon > 0$ there is $F \subset\subset \Lambda$ such that $\|\sum_{\lambda\in H} x_\lambda\| < \varepsilon$ for all $H \subset\subset \Lambda$ disjoint from F (in particular, $\|x_\lambda\| < \varepsilon$ for $\lambda \notin F$);*

(ii) *$\sum_{\lambda\in\Lambda} x_\lambda$ converges.*

Proof.

(i) $\Longrightarrow$ (ii): This follows since Cauchy nets in a Banach space converge, and

$$\Big\| \sum_{\lambda\in G_1} x_\lambda - \sum_{\lambda\in G_2} x_\lambda \Big\| \leqslant \Big\| \sum_{\lambda\in G_1\setminus G_2} x_\lambda \Big\| + \Big\| \sum_{\lambda\in G_2\setminus G_1} x_\lambda \Big\| \qquad (G_1, G_2 \subset\subset \Lambda).$$

(ii) $\Longrightarrow$ (i): This follows from the equality

$$\sum_{\lambda\in H} x_\lambda = \sum_{\lambda\in F\cup H} x_\lambda - \sum_{\lambda\in F} x_\lambda$$

for $H \subset\subset \Lambda \setminus F$.

Absolute convergence implies convergence for generalised series

Proposition

Let $\sum_{\lambda \in \Lambda} x_\lambda$ *be a generalised series in a Banach space* X*.*

(a) *If* $\sum_{\lambda \in \Lambda} x_\lambda$ *converges absolutely then it converges.*

(b) *If* $\sum_{\lambda \in \Lambda} x_\lambda$ *converges and* $X = \mathbb{C}$ *then it converges absolutely.*

Proof.
(a) Suppose that $\sum_{\lambda\in\Lambda}\|x_\lambda\|<\infty$ and let $\varepsilon>0$. Then there is $F\subset\subset\Lambda$ such that $\sum_{\lambda\in\Lambda\setminus F}\|x_\lambda\|<\varepsilon$ so that, for $H\subset\subset\Lambda$ disjoint from F,

$$\left\|\sum_{\lambda\in H}x_\lambda\right\|\leqslant\sum_{\lambda\in H}\|x_\lambda\|<\varepsilon.$$

This implies convergence of $\sum_{\lambda\in\Lambda}x_\lambda$.
(b) Suppose now that $X=\mathbb{C}$ and $\sum_{\lambda\in\Lambda}x_\lambda$ converges, to a say. By taking real and imaginary parts we may suppose that each $x_\lambda\in\mathbb{R}$. Set

$$A=\{\lambda\in\Lambda:x_\lambda\geqslant 0\}\quad\text{and}\quad B=\{\lambda\in\Lambda:x_\lambda<0\},$$

and choose $H\subset\subset\Lambda$ such that $\|\sum_{\lambda\in F}x_\lambda-a\|\leqslant 1$ when $H\subset F\subset\subset\Lambda$. Then, for $G\subset\subset A$,

$$\sum_{\lambda\in G}x_\lambda=\left|\sum_{\lambda\in H\cup G}x_\lambda-a+a-\sum_{\lambda\in H\setminus G}x_\lambda\right|\leqslant 1+|a|+\sum_{\lambda\in H}|x_\lambda|$$

so $\sum_{\lambda\in A}x_\lambda$ converges. Similarly $\sum_{\lambda\in B}x_\lambda$ converges. It follows that $\sum_{\lambda\in\Lambda}x_\lambda$ converges absolutely. □

Convergence does not imply absolute convergence

Example

In the Banach space c_0, the generalised series $\sum_{n\in\mathbb{N}} \frac{1}{n} e_n$ converges to $(1, \frac{1}{2}, \frac{1}{3}, \ldots) \in c_0$ but fails to converge absolutely, since the harmonic series is divergent.

Unconditionally convergent series need not converge absolutely.
In fact this is characteristic of infinite dimensionality.

Theorem (Dvoretzky and Rogers)

Let X be a Banach space. Then TFAE:

(i) *X is infinite-dimensional.*

(ii) *There is a convergent generalised series $\sum_{n\in\mathbb{N}} x_n$ in X which is not absolutely convergent.*

Fubini for absolutely convergent generalised series

Proposition

Let $\Lambda = \Gamma \times \Delta$ *for sets* Γ *and* Δ, *and let* $\sum_{\lambda \in \Lambda} a_\lambda$ *be an absolutely convergent generalised series in a Banach space* X. *Then*

(a) $\sum_{\delta \in \Delta} a_{\gamma,\delta}$ *is absolutely convergent for each* $\gamma \in \Gamma$*;*

(b) $\sum_{\gamma \in \Gamma} (\sum_{\delta \in \Delta} a_{\gamma,\delta})$ *is absolutely convergent;*

(c) $\sum_{\gamma \in \Gamma} \sum_{\delta \in \Delta} a_{\gamma,\delta} = \sum_{\lambda \in \Lambda} a_\lambda$.

The proof

Proof.

(a) and (b) are straightforward consequences of the definitions.

(c) Set $s = \sum_{\lambda \in \Lambda} a_\lambda$ and $t = \sum_{\gamma \in \Gamma} \sum_{\delta \in \Delta} a_{\gamma,\delta}$, let $\varepsilon > 0$ and successively choose

$$H_0 \subset\subset \Lambda \text{ such that } \left\| \sum_{\lambda \in H} a_\lambda - s \right\| < \varepsilon \text{ when } H_0 \subset H \subset\subset \Lambda;$$

$$G_0 \subset\subset \Gamma \text{ and } D_0 \subset\subset \Delta \text{ such that } G_0 \times D_0 \supset H_0;$$

$$G_1 \subset\subset \Gamma \text{ such that } G_1 \supset G_0 \text{ and } \left\| \sum_{\gamma \in G_1} \sum_{\delta \in \Delta} a_{\gamma,\delta} - t \right\| < \varepsilon, \text{ and}$$

$$D_1 \subset\subset \Delta \text{ such that } D_1 \supset D_0 \text{ and } \left\| \sum_{\delta \in \Delta \setminus D_1} a_{\gamma,\delta} \right\| < \frac{\varepsilon}{n} \text{ where } n = \# G_1.$$

Then

$$\|s - t\| \leqslant \left\| s - \sum_{\lambda \in G_1 \times D_1} a_\lambda \right\| + \left\| \sum_{\gamma \in G_1} \sum_{\delta \in \Delta \setminus D_1} a_{\gamma,\delta} \right\| + \left\| \sum_{\gamma \in G_1} \sum_{\delta \in \Delta} a_{\gamma,\delta} - t \right\| < 3\varepsilon,$$

and (c) follows. □