Recap of the last lecture

- **Theorem** (Constant Rank Theorem). Let $U \subset \mathbb{R}^m$ be an open subset and $u : U \rightarrow \mathbb{R}^n$ be a smooth map whose differential has constant rank $k$ in a neighbourhood of a point $p \in U$. Then there exist diffeomorphisms $F$ and $G$ defined in neighbourhoods of points $p \in \mathbb{R}^m$ and $u(p) \in \mathbb{R}^n$ respectively such that

\[
F(p) = 0 \in \mathbb{R}^m, \quad G(u(p)) = 0 \in \mathbb{R}^n, \quad \text{and} \quad G \circ u \circ F^{-1}(x^1, \ldots, x^m) = (x^1, \ldots, x^k, 0, \ldots, 0). \quad (\ast)
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  \]

- **Proof.** See [Conlon, Sect. 2.4].
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Plan of this lecture – consequences of the Constant Rank Theorem:
Theorem (Constant Rank Theorem). Let $U \subset \mathbb{R}^m$ be an open subset and $u : U \to \mathbb{R}^n$ be a smooth map whose differential has constant rank $k$ in a neighbourhood of a point $p \in U$. Then there exist diffeomorphisms $F$ and $G$ defined in neighbourhoods of points $p \in \mathbb{R}^m$ and $u(p) \in \mathbb{R}^n$ respectively such that

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Plan of this lecture – consequences of the Constant Rank Theorem:
- level sets as submanifolds: Regular Value Theorem and its generalisations;
Recap of the last lecture

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Plan of this lecture – consequences of the Constant Rank Theorem:

- level sets as submanifolds: Regular Value Theorem and its generalisations;
- notions of immersion and embedding;
- embeddings and submanifolds.
Corollary **. Let \( u : M \to N \) be a smooth map between smooth manifolds of dimensions \( m \) and \( n \) respectively. Suppose that the differential \( D_p u \) has constant rank \( k \) in a neighbourhood of \( u^{-1}(q) \) for some \( q \in N \). Then the pre-image \( u^{-1}(q) \subset M \) is a smooth submanifold of dimension \( m - k \).
Corollary. Let $u : M \to N$ be a smooth map between smooth manifolds of dimensions $m$ and $n$ respectively. Suppose that the differential $D_p u$ has constant rank $k$ in a neighbourhood of $u^{-1}(q)$ for some $q \in N$. Then the pre-image $u^{-1}(q) \subseteq M$ is a smooth submanifold of dimension $m - k$.

Remark: the classical regular value theorem is a partial case of this statement, when $m \geq n$, and rank $D_p u = n$ for any $p \in u^{-1}(q)$. 
Corollary **. Let \( u : M \to N \) be a smooth map between smooth manifolds of dimensions \( m \) and \( n \) respectively. Suppose that the differential \( D_p u \) has constant rank \( k \) in a neighbourhood of \( u^{-1}(q) \) for some \( q \in N \). Then the pre-image \( u^{-1}(q) \subset M \) is a smooth submanifold of dimension \( m - k \).

**Proof.** By Prop. 3 (see Lecture 3), for a proof it is sufficient to show that each \( p_0 \in u^{-1}(q) \) is contained in a chart \( (V_0, \psi_0) \) on \( M \) such that

\[
\psi_0(V_0 \cap u^{-1}(q)) = \psi_0(V_0) \cap \mathbb{R}^{m-k},
\]

where we view \( \mathbb{R}^{m-k} \) as a subspace of \( \mathbb{R}^m \). For the argument below we choose an embedding such that \( \mathbb{R}^{m-k} \simeq \{(0, \ldots, 0, x^{k+1}, \ldots, x^n, 0, \ldots, 0)\} \).
Corollary. Let $u : M \to N$ be a smooth map between smooth manifolds of dimensions $m$ and $n$ respectively. Suppose that the differential $D_p u$ has constant rank $k$ in a neighbourhood of $u^{-1}(q)$ for some $q \in N$. Then the pre-image $u^{-1}(q) \subset M$ is a smooth submanifold of dimension $m - k$.

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Given arbitrary charts $(V, \psi)$ and $(W, \phi)$ on $M$ and $N$, containing $p_0 \in u^{-1}(q)$ and $q$ respectively, we can apply the constant rank theorem to the local representation of $u$, that is the map $\phi \circ u \circ \psi^{-1}$. Making $V$ and $W$ smaller, if necessary, we can assume that there exist diffeomorphisms $F : \psi(V) \to \tilde{V} \subset \mathbb{R}^m$ and $G : \phi(W) \to \tilde{W} \subset \mathbb{R}^n$ such that $(G \circ \phi) \circ u \circ (F \circ \psi)^{-1}(x^1, \ldots, x^m) = (x^1, \ldots, x^k, 0, \ldots, 0)$. 


"Generalised" Regular Value Theorem

Corollary ∗∗. Let \( u : M \to N \) be a smooth map between smooth manifolds of dimensions \( m \) and \( n \) respectively. Suppose that the differential \( D_p u \) has constant rank \( k \) in a neighbourhood of \( u^{-1}(q) \) for some \( q \in N \). Then the pre-image \( u^{-1}(q) \subset M \) is a smooth submanifold of dimension \( m - k \).

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Given arbitrary charts \((V, \psi)\) and \((W, \phi)\) on \( M \) and \( N \), containing \( p_0 \in u^{-1}(q) \) and \( q \) respectively, we can apply the constant rank theorem to the local representation of \( u \), that is the map \( \phi \circ u \circ \psi^{-1} \). Making \( V \) and \( W \) smaller, if necessary, we can assume that there exist diffeomorphisms \( F : \psi(V) \to \tilde{V} \subset \mathbb{R}^m \) and \( G : \phi(W) \to \tilde{W} \subset \mathbb{R}^n \) such that \( (G \circ \phi) \circ u \circ (F \circ \psi)^{-1}(x^1, \ldots, x^m) = (x^1, \ldots, x^k, 0, \ldots, 0) \).

We claim that \((V, F \circ \psi)\) is the desired chart. Indeed, we have

\[
(G \circ \phi) \circ u \circ (F \circ \psi)^{-1}(F \circ \psi(V \cap u^{-1}(q))) = (G \circ \phi) \circ u(V \cap u^{-1}(q)) = 0,
\]

since \( G \circ \phi(q) = 0 \). From the form (\( * \)) we conclude that \( x^1 = \cdots = x^k = 0 \), and \( x^{k+1}, \ldots, x^m \) are constrained only by the condition that \((x^1, \ldots, x^m)\) lies in the open set \( F \circ \psi(V) \).
Corollary. Let \( u : M \to N \) be a smooth map between smooth manifolds of dimensions \( m \) and \( n \) respectively. Suppose that the differential \( D_p u \) has constant rank \( k \) in a neighbourhood of \( u^{-1}(q) \) for some \( q \in N \). Then the pre-image \( u^{-1}(q) \subset M \) is a smooth submanifold of dimension \( m - k \).

Proof. By Prop. 3 (see Lecture 3), for a proof it is sufficient to show that each \( p_0 \in u^{-1}(q) \) is contained in a chart \((V_0, \psi_0)\) on \( M \) such that

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\psi_0(V_0 \cap u^{-1}(q)) = \psi_0(V_0) \cap \mathbb{R}^{m-k},
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Given arbitrary charts \((V, \psi)\) and \((W, \phi)\) on \( M \) and \( N \), containing \( p_0 \in u^{-1}(q) \) and \( q \) respectively, we can apply the constant rank theorem to the local representation of \( u \), that is the map \( \phi \circ u \circ \psi^{-1} \). Making \( V \) and \( W \) smaller, if necessary, we can assume that there exist diffeomorphisms \( F : \psi(V) \to \tilde{V} \subset \mathbb{R}^m \) and \( G : \phi(W) \to \tilde{W} \subset \mathbb{R}^n \) such that \((G \circ \phi) \circ u \circ (F \circ \psi)^{-1}(x^1, \ldots, x^m) = (x^1, \ldots, x^k, 0, \ldots, 0)\).

Conversely, if \( x(p) = (x^1, \ldots, x^m) \) lies in \( F \circ \psi(V) \cap \mathbb{R}^{m-k} \), then \( x^1 = \cdots = x^k = 0 \), and we conclude that \( G \circ \phi(u(p)) = 0 \), and hence \( u(p) = q \). Thus, relation (**) indeed holds with \( V_0 = V \) and \( \psi_0 = F \circ \psi \). \( \square \)
Example 8 (revisited again). The unit sphere $S^n \subset \mathbb{R}^{n+1}$. 
**Example 8 (revisited again).** The unit sphere $S^n \subset \mathbb{R}^{n+1}$.

Consider the function $u : \mathbb{R}^{n+1} \to \mathbb{R}$, given by $u(x^1, \ldots, x^{n+1}) = \sum_{i=1}^{n+1} x_i^2$. Then the unit sphere $S^n$ is precisely the level set $u^{-1}(1)$. By Corollary $\star\star$ it is sufficient to check that $\text{rank}(D_x u)$ equals one for any $x \in \mathbb{R}^{n+1} \setminus \{0\}$. Differentiating, we obtain

$$D_x u = (2x^1, \ldots, 2x^{n+1}) \neq 0 \quad \text{when} \ x \in \mathbb{R}^{n+1} \setminus \{0\}.$$
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Example 10. $SL(n, \mathbb{R}) = \{A \text{ is an } n \times n\text{-matrix} : \det A = 1\} \subset \mathbb{R}^{n \times n}$. 
Example 8 (revisited again). The unit sphere \( S^n \subset \mathbb{R}^{n+1} \).
Consider the function \( u : \mathbb{R}^{n+1} \to \mathbb{R} \), given by \( u(x^1, \ldots, x^{n+1}) = \sum_{i=1}^{n+1} x_i^2 \). Then the unit sphere \( S^n \) is precisely the level set \( u^{-1}(1) \). By Corollary \( \star \star \) it is sufficient to check that \( \text{rank}(D_xu) \) equals one for any \( x \in \mathbb{R}^{n+1}\{0\} \). Differentiating, we obtain
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\]

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Consider the function \( F : \mathbb{R}^{n \times n} \to \mathbb{R}, F(A) = \det A \). Note that \( F \) is smooth, and \( SL(n, \mathbb{R}) = F^{-1}(1) \).
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Consider the function $u : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$, given by $u(x^1, \ldots, x^{n+1}) = \sum_{i=1}^{n+1} x_i^2$. Then the unit sphere $S^n$ is precisely the level set $u^{-1}(1)$. By Corollary \(*\) it is sufficient to check that rank($D_x u$) equals one for any $x \in \mathbb{R}^{n+1}\{0\}$. Differentiating, we obtain

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Consider the function $F : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$, $F(A) = \det A$. Note that $F$ is smooth, and $SL(n, \mathbb{R}) = F^{-1}(1)$. Recall (see a Linear Algebra course) that

$$\det(E + tA) = 1 + p_1(A)t + p_2(A)t^2 + \cdots + p_n(A)t^n,$$

where $p_i(A)$ is the sum of all principal $i \times i$-minors of $A$, e.g. $p_1(A) = \text{trace}A$, $p_n(A) = \det A$. 
**Examples**

**Example 8 (revisited again).** The unit sphere \( S^n \subset \mathbb{R}^{n+1} \).

Consider the function \( u : \mathbb{R}^{n+1} \to \mathbb{R} \), given by \( u(x^1, \ldots, x^{n+1}) = \sum_{i=1}^{n+1} x_i^2 \). Then the unit sphere \( S^n \) is precisely the level set \( u^{-1}(1) \). By Corollary \( \star \star \) it is sufficient to check that \( \text{rank}(D_x u) \) equals one for any \( x \in \mathbb{R}^{n+1}\setminus\{0\} \). Differentiating, we obtain

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**Example 10.** \( SL(n, \mathbb{R}) = \{ A \text{ is an } n \times n \text{-matrix } : \det A = 1 \} \subset \mathbb{R}^{n \times n} \).

Consider the function \( F : \mathbb{R}^{n \times n} \to \mathbb{R}, F(A) = \det A \). Note that \( F \) is smooth, and \( SL(n, \mathbb{R}) = F^{-1}(1) \). Recall (see a Linear Algebra course) that

\[
\det(E + tA) = 1 + p_1(A)t + p_2(A)t^2 + \cdots + p_n(A)t^n,
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where \( p_i(A) \) is the sum of all principal \( i \times i \)-minors of \( A \), e.g. \( p_1(A) = \text{trace}A \), \( p_n(A) = \det A \).

Now let \( A \in GL(n, \mathbb{R}) = \mathbb{R}^{n \times n}\setminus \det^{-1}(0) \). We would like to understand the differential \( D_A F : \mathbb{R}^{n \times n} \to \mathbb{R} \). By the Chain Rule, we obtain

\[
D_A F(B) = \left. \frac{d}{dt} \right|_{t=0} \det(A + tB) = \det A \left. \frac{d}{dt} \right|_{t=0} \det(E + tA^{-1}B) = \det A \cdot \text{trace}(A^{-1}B).
\]

In particular, \( D_A F(A) = \det A \cdot \text{trace}(E) = n \det A \neq 0 \), where we used that \( \det A \neq 0 \). Thus, the rank of \( D_A F \) equals one, and by Corollary \( \star \star \), we conclude that \( SL(n, \mathbb{R}) \) is a submanifold of \( \mathbb{R}^{n \times n} \) of dimension...
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Consider the function $u : \mathbb{R}^{n+1} \to \mathbb{R}$, given by $u(x^1, \ldots, x^{n+1}) = \sum_{i=1}^{n+1} x_i^2$. Then the unit sphere $S^n$ is precisely the level set $u^{-1}(1)$. By Corollary ⋆⋆ it is sufficient to check that $\text{rank}(D_xu)$ equals one for any $x \in \mathbb{R}^{n+1}\{0\}$. Differentiating, we obtain
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Example 10. $SL(n, \mathbb{R}) = \{A \text{ is an } n \times n\text{-matrix : } \det A = 1\} \subset \mathbb{R}^{n \times n}$.
Consider the function $F : \mathbb{R}^{n \times n} \to \mathbb{R}$, $F(A) = \det A$. Note that $F$ is smooth, and $SL(n, \mathbb{R}) = F^{-1}(1)$. Recall (see a Linear Algebra course) that
$$\det(E + tA) = 1 + p_1(A)t + p_2(A)t^2 + \cdots + p_n(A)t^n,$$
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Now let $A \in GL(n, \mathbb{R}) = \mathbb{R}^{n \times n}\setminus \det^{-1}(0)$. We would like to understand the differential $D_AF : \mathbb{R}^{n \times n} \to \mathbb{R}$. By the Chain Rule, we obtain
$$D_AF(B) = \left. \frac{d}{dt} \right|_{t=0} \det(A + tB) = \det A \left. \frac{d}{dt} \right|_{t=0} \det(E + tA^{-1}B) = \det A \cdot \text{trace}(A^{-1}B).$$
In particular, $D_AF(A) = \det A \cdot \text{trace}(E) = n \det A \neq 0$, where we used that $\det A \neq 0$. Thus, the rank of $D_AF$ equals one, and by Corollary ⋆⋆, we conclude that $SL(n, \mathbb{R})$ is a submanifold of $\mathbb{R}^{n \times n}$ of dimension $n^2 - 1$. 
Anti-example. The cone $C = \{(x, y, z) : x^2 + y^2 = z^2\} \subset \mathbb{R}^3$. 
Examples

Anti-example. The cone \( C = \{(x, y, z) : x^2 + y^2 = z^2 \} \subset \mathbb{R}^3 \).

Consider the function \( u : \mathbb{R}^{n+1} \to \mathbb{R} \), given by \( u(x, y, z) = x^2 + y^2 - z^2 \). Then the cone \( C \) is precisely the level set \( u^{-1}(0) \). Differentiating, we obtain

\[
D_p u = (2x, 2y, -2z) \neq 0 \quad \text{when } p \in \mathbb{R}^3 \setminus \{0\},
\]

and \( D_p u = 0 \), when \( p = 0 \).
Anti-example. The cone $C = \{(x, y, z) : x^2 + y^2 = z^2\} \subset \mathbb{R}^3$. Consider the function $u : \mathbb{R}^{n+1} \to \mathbb{R}$, given by $u(x, y, z) = x^2 + y^2 - z^2$. Then the cone $C$ is precisely the level set $u^{-1}(0)$. Differentiating, we obtain

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Conclusion. Since $0 \in C$, Corollary * does not apply. (This is consistent with the fact that $C$ is not even a topological manifold: there is no neighbourhood of 0 in $C$ that is homeomorphic to an open subset in $\mathbb{R}^2$.)
Anti-example. The cone $C = \{(x, y, z) : x^2 + y^2 = z^2\} \subset \mathbb{R}^3$.
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Conclusion. Since $0 \in C$, Corollary ⋆⋆ does not apply. (This is consistent with the fact that $C$ is not even a topological manifold: there is no neighbourhood of $0$ in $C$ that is homeomorphic to an open subset in $\mathbb{R}^2$.)

However, Corollary ⋆⋆ applies to the set $C \setminus \{0\}$, which is a submanifold of $\mathbb{R}^3$. 
Examples

Anti-example. The cone $C = \{(x, y, z) : x^2 + y^2 = z^2\} \subset \mathbb{R}^3$.

Consider the function $u : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$, given by $u(x, y, z) = x^2 + y^2 - z^2$. Then the cone $C$ is precisely the level set $u^{-1}(0)$. Differentiating, we obtain

$$D_p u = (2x, 2y, -2z) \neq 0 \quad \text{when } p \in \mathbb{R}^3 \setminus \{0\},$$

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Conclusion. Since $0 \in C$, Corollary ⋆⋆ does not apply. (This is consistent with the fact that $C$ is not even a topological manifold: there is no neighbourhood of 0 in $C$ that is homeomorphic to an open subset in $\mathbb{R}^2$.)

However, Corollary ⋆ applies to the set $C \setminus \{0\}$, which is a submanifold of $\mathbb{R}^3$.

Exercise 11. Give an example of a non-constant smooth function $u : \mathbb{R}^3 \rightarrow \mathbb{R}$ and a real number $c$ such that the rank of $D_x u$ is not constant as $x$ ranges over $u^{-1}(c)$, but the level set $u^{-1}(c)$ is a smooth submanifold of $\mathbb{R}^3$. 
Anti-example. The cone \( C = \{(x, y, z) : x^2 + y^2 = z^2\} \subset \mathbb{R}^3 \).

Consider the function \( u : \mathbb{R}^{n+1} \to \mathbb{R} \), given by \( u(x, y, z) = x^2 + y^2 - z^2 \). Then the cone \( C \) is precisely the level set \( u^{-1}(0) \). Differentiating, we obtain

\[
D_p u = (2x, 2y, -2z) \neq 0 \quad \text{when } p \in \mathbb{R}^3 \setminus \{0\},
\]

and \( D_p u = 0 \), when \( p = 0 \).

**Conclusion.** Since \( 0 \in C \), Corollary ⋆⋆ does not apply. (This is consistent with the fact that \( C \) is not even a topological manifold: there is no neighbourhood of 0 in \( C \) that is homeomorphic to an open subset in \( \mathbb{R}^2 \).)

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**Exercise 11.** Give an example of a non-constant smooth function \( u : \mathbb{R}^3 \to \mathbb{R} \) and a real number \( c \) such that the rank of \( D_x u \) is not constant as \( x \) ranges over \( u^{-1}(c) \), but the level set \( u^{-1}(c) \) is a smooth submanifold of \( \mathbb{R}^3 \).

**Q&A pause.**
Definition. A smooth map $u: M \to N$ is called the *immersion* if for any $p \in M$ the differential $D_p u : T_p M \to T_{u(p)} N$ is injective (equivalently, $D_p u$ has a trivial kernel).
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Comments, examples, drawings; comparison with regular parametrised curves and surfaces.
**Immersions and embeddings**

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**Corollary.** Let $u : M \to N$ be an immersion. Then for any point $p \in M$ there exists a neighbourhood $U_p \ni p$ such that $u|_{U_p}$ is an embedding.

**Sketch of a proof.** Let $m$ and $n$ be dimensions of $M$ and respectively, $m \leq n$. Since $u$ is an immersion, we see that rank $D_p u = m$ for any point $p \in M$. Choosing charts $(U, \varphi)$ and $(V, \psi)$ on $M$ and $N$ such that $p \in U$ and $u(p) \in V$, we may apply the constant rank theorem to the local representation $\psi \circ u \circ \varphi^{-1}$. Thus, we obtain

$$(G \circ \psi) \circ u \circ (F \circ \varphi)^{-1}(x^1, \ldots, x^m) = (x^1, \ldots, x^m, 0, \ldots, 0),$$

where $F$ and $G$ are appropriate diffeomorphisms and $x = (x^1, \ldots, x^m)$ ranges in an open set $B$ such that $(F \circ \varphi)^{-1}(B) \subset U$. From the above we conclude that $(G \circ \psi) \circ u \circ (F \circ \varphi)^{-1}$ is a homeomorphism onto its image, and hence, so is the map $u$ restricted to $(F \circ \varphi)^{-1}(B)$. 

$\square$
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Corollary. Let \( u : M \to N \) be an immersion. Then for any point \( p \in M \) there exists a neighbourhood \( U_p \ni p \) such that \( u\|_{U_p} \) is an embedding.

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Exercise 12. Let \( u : M \to N \) be an injective immersion. Show that if \( M \) is compact, then \( u \) is an embedding.
Final questions
THE END