FUNCTIONAL ANALYSIS
via MAGIC

METRIC SPACES & FURTHER TOPOLOGY (Topic 3)

J. Martin Lindsay

Department of Mathematics and Statistics,

Lancaster University

LAST EDITED: 14.x.2020

Introduction

In this section we cover the elementary theory of metric spaces and some further topology. There is an emphasis on completeness and separability, whose marriage gives rise to the fruitful concept of Polish space. Compactness has simplifying equivalents in the metric space context through the concepts of total boundedness and sequential compactness. A topological measure of *smallness* called meagreness helps us, for example, to see differentiability of a continuous function as atypical. By means of the Baire Category Theorem we also see the impossibility of a Banach space having countable dimension as a vector space and obtain the powerful Uniform Boundedness Principle.

A key way in which topology and metric space theory meet in functional analysis is through metric spaces of bounded continuous (vector-valued) functions on a topological space. Separation and extension properties are important here, and these are covered along with Alexandroff's one-point compactification and the Stone-Cech compactification. The section ends with Stone's far-reaching generalisation of the classical Weierstrass Approximation Theorem.

Notation

For a subset A of a metric space E and $\delta > 0$,

$$A^{(\delta)} := \{x \in E : \ \mathrm{dist}(x, A) < \delta\} = d_A^{-1}([0, \delta[)$$

(see below for 'dist'). Thus, when E is a normed linear space, $A^{(\delta)} = A + B_\delta(0)$.

MS-1

- Isometry
- Cauchy condition; completeness
- Cantor's Intersection Theorem
- (Bounded) equivalent metrics
- Seminorms; Banach space

Isometry

Definition

A map f from a metric space (E, d) to a metric space (E', d') is an *isometry* if it satisfies

$$d'\big(f(x), f(y)\big) = d(x, y) \quad \text{for all } x, y \in E.$$

If f is surjective then the metric spaces are said to be *isometric*.

Consequences

- Isometries are injective and continuous.
- The inverse of a surjective isometry is an isometry.
- $\mathrm{Isom}(E, d)$ is a subgroup of $\mathrm{Homeo}(E, \tau_d)$, where τ_d denotes the induced topology.

Isometric spaces are *metrically* indistinguishable.

Diameter, boundedness and distance from a set

Definition

Let (E, d) be a metric space and let $A \subset E$, $A \neq \emptyset$.

- The *diameter* of A is defined by

$$\operatorname{diam} A := \sup \left\{ d(x, y) : x, y \in A \right\},$$

  and A is said to be *bounded* if $\operatorname{diam} A < \infty$.

- For an element z of E, its *distance from A* is given by

$$\operatorname{dist}(z, A) := \inf \left\{ d(z, a) : a \in A \right\}.$$

Consequences

- $\operatorname{diam} A \in [0, \infty]$ and $\operatorname{dist}(z, A) \in [0, \infty[$.
- There are sequences $(a_n), (b_n)$ and (c_n) in A satisfying

$$\forall_{n \in \mathbb{N}} \ \ d(z, a_n) < \operatorname{dist}(z, A) + 1/n, \qquad \text{resp. } \forall_{n \in \mathbb{N}} \ \ d(b_n, c_n) > \operatorname{diam} A - 1/n.$$

- $\left\{ z \in E : \ \operatorname{dist}(z, A) = 0 \right\} = \overline{A}$.

Definition

Let (E, d) be a metric space. A sequence (x_n) in E is *Cauchy* if

$$\forall_{r>0} \, \exists_{N \in \mathbb{N}} \, \forall_{n,m \geq N} \quad d(x_n, x_m) < r.$$

A subset $C \subset E$ is *complete* if any Cauchy sequence (x_n) in C converges with $\lim x_n \in A$.

Consequences

Let $A \subset C \subset E$, where C is complete. Then

- C is closed;
- A is complete if and only if it is closed;
- Any isometric image of C is complete.

Example

The metric spaces $]0, 1[$ and $\mathbb{R}$ are homeomorphic, yet $\mathbb{R}$ is complete whilst $]0, 1[$ is clearly not.

Thus completeness is *not* a topological property.

Lemma

Let (x_n) be a sequence in a metric space (E, d).
(a) If (x_n) is convergent then it is Cauchy.
(b) If (x_n) is Cauchy then

- *(x_n) is bounded;*
- *$\big(d(a, x_n)\big)$ converges in $\mathbb{R}$, for each $a \in E$;*
- *if (x_n) also has a convergent subsequence then (x_n) itself converges.*

Proof.
Exercise. (Recall the equivalent form for the triangle inequality:

$$\big|d(x, y) - d(x, z)\big| \leq d(y, z)).$$

□

Proposition (Cantor's Intersection Theorem)

Let E be a complete metric space, and let $F = \bigcap_{n \in \mathbb{N}} F_n$ where (F_n) is a decreasing sequence of nonempty closed subsets of E. If $\operatorname{diam} F_n \to 0$ then F is a singleton set.

Proof.

For each $n \in \mathbb{N}$ choose $x_n \in F_n$. Since $\operatorname{diam} F_n \to 0$ it follows that (x_n) is Cauchy and so convergent. Let $a = \lim x_n$. Then, for all $N \in \mathbb{N}$, since $x_n \in F_N$ for all $n \geq N$ and F_N is closed it follows that $a \in F_N$. Thus $a \in F$. This shows that F is nonempty. Since $\operatorname{diam} F \leq \inf_{n \in \mathbb{N}} \operatorname{diam} F_n = 0$, F can contain no more than one element. □

Exercise.

Show that the theorem fails if the condition $\operatorname{diam} F_n \to 0$ is dropped.

Definition (Equivalent metrics)

Two metrics on a set S are *equivalent* if they induce the same topology on S.

Lemma (Equivalence of metrics by sequences)

Let d and ρ be metrics on a set S. Then TFAE:

(i) *d and ρ are equivalent metrics;*

(ii) *for any sequence (x_n) in S and element $a \in S$,*

$$d(x_n, a) \to 0 \quad \Leftrightarrow \quad \rho(x_n, a) \to 0.$$

Proof.

Exercise. $\square$

Proposition (Equivalent bounded metric)

Let (E, d) be a metric space. Then there is an equivalent metric on E with respect to which E is bounded.

Proof.

Set

$$\widetilde{d}(x, y) = f\big(d(x, y)\big)$$

where

$$f : [0, \infty[\to [0, \infty[, \quad t \mapsto t/(1 + t) \text{ or } \min\{t, 1\}.$$

Exercise. Check that $\widetilde{d}$ is a metric and that it is equivalent to d. More generally, explore sufficient conditions on a function $f : [0, \infty[\to [0, \infty[$ for d and $f \circ d$ to be equivalent metrics for *every metric d*. $\square$

Definition

A *seminorm* on a vector space V over $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, is a map $p : V \to \mathbb{R}$ satisfying

$$\text{(Ni)} \quad p(u+v) \leq p(u) + p(v)$$
$$\text{(Nii)} \quad p(\lambda v) = |\lambda| p(v)$$

for $u, v, \in V$ and $\lambda \in \mathbb{K}$. It is a *norm* if further

$$\text{(Niii)} \quad p(v) = 0 \Rightarrow v = 0.$$

Norms are almost always denoted $\| \cdot \|$.

Consequences

- $\big|p(v) - p(u)\big| \leq p(v-u)$,
- $p(0) = 0$ and
- $p(v) \geq 0$,

for all $u, v \in V$.

Proposition

Let p be a seminorm on a vector space V.

- The prescription
$$d(u, v) := p(v - u)$$
defines a pseudometric on V which is translation invariant: for all $u, v, w \in V$,
$$d(u + w, v + w) = d(u, v)$$
and is a *metric* if and only if p is a norm.
- Addition, scalar multiplication and the seminorm are all continuous w.r.t. the topology induced by the pseudometric (respectively $V \times V \to V$, $\mathbb{K} \times V \to V$, $V \to \mathbb{R}$).

Proof.

Exercise. [HINT: Use the sequence characterisation of continuity.] □

Definition

A normed linear space $(X, \|\cdot\|)$, here abbreviated to NLS, is called a *Banach space* if it is complete w.r.t. the induced metric.

MS-2

- Metric spaces of (continuous) bounded functions
- $l^\infty(S)$ and $l^\infty(S; X)$
- Isometric embedability of every metric space in a Banach space
- Uniform continuity and the Lipschitz property

Proposition

Let S be a set and (E,d) a metric space.

(a)

$$d_{\sup}(f,g) := \sup\{d(f(s),g(s)) : s \in S\}$$

defines a metric on

$$\mathcal{F}_b(S;(E,d)) := \{f \in \mathcal{F}(S;E) : f \text{ is bounded}\},$$

where 'bounded' for a function f means: $\operatorname{diam} f(S) < \infty$.

(b) The resulting metric space is complete if (E,d) is.

(c) If S is a topological space then

$$C_b(S;(E,d)) := \{f \in \mathcal{F}_b(S;(E,d)) : f \text{ is continuous}\}$$

is a closed subset of $\mathcal{F}_b(S;(E,d))$ — in particular it is a complete metric space, in the sup-metric, if (E,d) is complete.

Remark

Convergence with respect to the sup-metric is called *uniform convergence*.

Proof.

(a) Exercise.

(b) Suppose that (E,d) is complete. Let (f_n) be a Cauchy sequence in $\mathcal{F}_b(S;(E,d))$. Then, for each $s \in S$, $\big(f_n(s)\big)$ is Cauchy in E and so convergent. Therefore the prescription $s \mapsto \lim f_n(s)$ defines a function $S \to E$. It remains to show that

(i) f is bounded;

(ii) $f_n \to f$ uniformly.

This is left as an exercise.

(c) The claim here is a familiar one, namely that a uniform limit of continuous functions is continuous. This is left as an exercise too.

□

Corollary

For a set S and NLS X, $\mathcal{F}_b(S;X)$ is a linear subspace of the vector space $\mathcal{F}(S;X)$ on which

$$\|f\|_{\sup} := \sup\left\{\|f(s)\|_X : s \in S\right\}$$

defines a norm. The resulting NLS is denoted $l^\infty(S;X)$ — it is a Banach space if X is.

Proof.

The subspace and norm properties are straight-forward. The rest follows easily from the Proposition on $d_{\sup}$, since the induced metric is $d_{\sup}$. □

Notation

The Banach spaces $l^\infty(S;\mathbb{C})$, $l^\infty(S;\mathbb{R})$, $l^\infty(\mathbb{N};\mathbb{C})$ and $l^\infty(\mathbb{N};\mathbb{R})$ are abbreviated respectively to $l^\infty(S)$, $l^\infty_{\mathbb{R}}(S)$, l^∞ and $l^\infty_{\mathbb{R}}$. The latter being spaces of bounded sequences with sup-norm.

$\|\cdot\|_{\sup}$ is very often denoted $\|\cdot\|_\infty$.

Every metric space embeds isometrically in a Banach space

Theorem

Let (E, d) be a metric space. Then there is an isometry from E into a real Banach space.

Proof. Let X be the real Banach space $l^{\infty}_{\mathbb{R}}(E)$. For each $a \in E$ define the function

$$d_a : E \to \mathbb{R}, \quad x \mapsto d(a, x).$$

Since $\left|d(a, x) - d(a', x)\right| \leq d(a, a')$ for all $x, a, a' \in E$, with equality if $x = a'$, it follows that, for all $a, a' \in E$,

$$(d_a - d_{a'}) \in X \text{ and } \|d_a - d_{a'}\|_{\sup} = d(a, a')$$

(subtraction in the real vector space $\mathcal{F}_{\mathbb{R}}(S)$). Fix $a_0 \in E$ and define a map

$$j : E \to X, \quad a \mapsto (d_a - d_{a_0}).$$

Then j is isometric since, for all $a, b \in E$,

$$\left\|j(a) - j(b)\right\|_{\infty} = \|d_a - d_{\mathrm{b}}\|_{\infty} = d(a, b).$$

Uniform continuity and the Lipschitz property

Definition

Let $f \in \mathcal{F}(E; E')$ for metric spaces (E, d) and (E', d'). Then f is *uniformly continuous*, if

$$\forall_{\varepsilon>0} \exists_{\delta>0} \forall_{x,y\in E} \; \big[d(x, y) < \delta \Rightarrow d'\big(f(x), f(y)\big) < \varepsilon\big];$$

and f is *Lipschitz* (*continuous*) if

$$\exists_{L\geq 0} \forall_{x,y\in E} \; d'\big(f(x), f(y)\big) \leq L \; d(x, y),$$

in which case its *Lipschitz constant* is

$$\operatorname{Lip} f := \sup\big\{d'(f(x), f(y))\big/d(x, y) : x \neq y\big\}.$$

If $\operatorname{Lip} f < 1$ then f is called a *strict contraction*.

Consequences

- Isometric $\Rightarrow$ Lipschitz $\Rightarrow$ uniformly continuous.
- $\operatorname{Lip} f > 0$ unless f is constant.
- (x_n) Cauchy in E, f uniformly continuous $\Rightarrow$ $\big(f(x_n)\big)$ Cauchy.

Proposition

Let E be a metric space and X a NLS, and let $h = f + \lambda g$ where $f, g \in \mathcal{F}(E;X)$ and $\lambda \in \mathbb{K}$. If f and g are continuous (resp. uniformly continuous or Lipschitz) then so is h. Moreover,

- $\operatorname{Lip}(f+g) \leq \operatorname{Lip} f + \operatorname{Lip} g$;
- $\operatorname{Lip}(\lambda f) = |\lambda| \operatorname{Lip} f$;
- $\operatorname{Lip} f > 0$ unless f is constant.

Proof.

Exercise. □

d_A is Lipschitz; automatic uniformity of continuity from compactness

Example (d_A is Lipschitz)

For a subset A of a metric space E, the 'distance from A' function

$$d_A : E \to \mathbb{R}, \quad x \mapsto \mathrm{dist}(x, A)$$

is Lipschitz, moreover $\mathrm{Lip}\, d_A = 1$ (unless A is dense in E).

Proof.
Exercise. □

Proposition (Automatic boundedness and uniform continuity)

Let $f \in C(K; E)$ for a compact metric space K and metric space E. Then f is bounded and uniformly continuous.

Proof.
Since $f(K)$ is a compact subset of E it is bounded (let $a \in A$ then the open cover $\bigcup_{n\in\mathbb{N}} B_n^E(a)$ has a finite subcover). Uniform continuity is left as an exercise. □

Proposition (Contraction Mapping Theorem)

Let E be a complete metric space and let τ be a strict contraction on E (that is $\operatorname{Lip}\tau < 1$). Then there is a unique point $a \in E$ such that $\tau(a) = a$. Moreover, letting τ^n denote the n-fold composition $\tau \circ \cdots \circ \tau$,

$$\tau^n(x) \to a \quad \text{for all } x \in E.$$

Proof.

Let $z \in E$ and set $z_n = \tau^n(z)$ for $n = 1, 2, \ldots$. Then

$$d(z_n, z_{n+1}) \le Ld(z_{n-1}, z_n) \le \ldots \le L^n d\big(z, \tau(z)\big) \text{ for } n \in \mathbb{N}$$

where $L := \operatorname{Lip} \tau$ and so, by the triangle inequality,

$$\begin{aligned} d(z_n, z_{n+p}) &\le d(z_n, z_{n+1}) + \cdots + d(z_{n+p-1}, z_{n+p}) \\ &\le L^n d\big(z, \tau(z)\big) + \cdots + L^{n+p-1} d\big(z, \tau(z)\big) \le L^n (1 - L)^{-1} d\big(z, \tau(z)\big). \end{aligned}$$

Thus (z_n) is Cauchy and so convergent, to a say. For $x \in E$,

$$d\big(\tau^n(z), \tau^n(x)\big) \le L^n \tau(z, x) \to 0,$$

so $\tau^n(x) \to a$ too. In particular, if $b \in E$ satisfies $\tau(b) = b$ then $\tau^n(b) = b$ for all $n \in \mathbb{N}$ so $b = a$. Finally, since τ is continuous,

$$\tau(a) = \tau(\lim x_n) = \lim \tau(x_n) = \lim x_{n+1} = a.$$

□

Example (Hausdorff metrics)

For $n \in \mathbb{N}$, set

$$\mathcal{K}_n := \{K \subset \mathbb{R}^n : K \text{ is nonempty and compact}\},$$

and for $J, K \in \mathcal{K}_n$, set

$$d(J,K) := \inf\left\{\delta > 0 : J^{(\delta)} \supset K \text{ and } K^{(\delta)} \supset J\right\}.$$

Then

- d is a complete metric on $\mathcal{K}_n$;
- the map $\tau : \mathcal{K}_1 \to \mathcal{K}_1$, $K \mapsto \frac{1}{3}K \cup \left(\frac{1}{3}K + \frac{2}{3}\right)$, is a strict contraction $\left(\operatorname{Lip}\tau = 1/3\right)$;
- the unique 'fixed point' of the map τ is Cantor's Middle Thirds set C;
- in particular, $\tau^n\big(\{0\}\big) \to C$ as $n \to \infty$,
  cf. the more familiar fact that $\tau^n\big([0,1]\big) \to C$ as $n \to \infty$.

Proof.
Exercise. □

MS-3

- Sequential compactness and total boundedness
- Compactness of closed balls $\equiv$ finite dimensionality
- Arzela-Ascoli

Definition

Let (E, d) be a metric space and let $A \subset E$. Then A is

- *sequentially compact* if every sequence in A has a subsequence which converges to a point in A;
- *totally bounded* if, for every $r > 0$ there is a finite number of open balls of radius r which cover A:

$$\forall_{r>0}\, \exists_{n\in\mathbb{N}, x_1,\dots,x_n \in E} : \quad \bigcup_{i=1}^{n} B_r(x_i) \supset A.$$

Consequences

- Sequentially compact sets are complete.
- Totally bounded sets are bounded and separable.

Example

In a discrete metric space, any infinite set is bounded but not totally bounded.

Theorem (Characterisation of compactness in a metric space)

Let $A \subset E$ for a metric space (E, d). Then TFAE:

(i) *A is compact;*

(ii) *A is sequentially compact;*

(iii) *A is complete and totally bounded.*

Theorem (F. Riesz)

Let B be the closed unit ball of a NLS X. Then TFAE:

(i) *B is compact;*

(ii) $\dim X < \infty$.

Corollary

Let E_i be a metrisable space and let $K_i \subset E_i$ be compact, for $i = 1, \ldots, n$. Then $K_1 \times \cdots \times K_n$ is compact in the product topology.

Proof.

Exercise. $\big[$HINT: If d_i metrises the topology of E_i, for $i = 1, \ldots, n$, then

$$(x, y) \mapsto \sum_{i=1}^{n} d_i(x_i, y_i)$$

metrises the product topology. Show that $K_1 \times \cdots \times K_n$ is sequentially compact w.r.t. this metric.$\big]$ □

Proposition

Let (E, d) be a metric space, let $K \subset E$ be compact and let $a \in E$. Then

(a) $\exists_{x,y \in K} : \quad d(x, y) = \operatorname{diam} K$.

(b) $\exists_{z \in K} : \quad d(a, z) = \operatorname{dist}(a, K)$.

Proof.

Exercise. □

Theorem (Arzela-Ascoli)

Let K be a compact metric space. For a subset S of the Banach space $C(K) := C(K;\mathbb{C})$, TFAE:

(i) *S is totally bounded;*

(ii) *S is bounded and equicontinuous.*

Equicontinuous means

$$\forall_{a\in K,\varepsilon>0}\,\exists_{\delta>0}\,\forall_{x\in K,f\in R}\;\; \big[d(x,a)<\delta \Rightarrow \big|f(x)-f(a)\big|<\varepsilon\big].$$

Remark

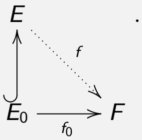
Equicontinuity for $S \subset C(K)$ implies *uniform equicontinuity* (with obvious definition!).

MS-4

- Uniformly continuous extension
- Metric space completion
- Compactness via completeness
- Complete metrisability

Theorem (Uniformly continuous extension)

Let $f_0 : E_0 \to F$ be a uniformly continuous map from a dense subset E_0 of a metric space E into a complete metric space F. Then there is a unique continuous map $f : E \to F$ extending f_0:



Moreover, f is uniformly continuous, and

- *f is Lipschitz if f_0 is, with the same Lipschitz constant,*
- *f is isometric if f_0 is.*

Proof.

Uniqueness: This follows from the Identity Theorem.
Existence: For any such extension f, and any $x \in E$,

(a) by density, there is a sequence (x_n) in E_0 converging to x;

(b) by continuity, $f(x) = \lim f(x_n) = \lim f_0(x_n)$.

The strategy is therefore clear — we seek to *define* f by $f(x) = \lim_{n\to\infty} f_0(x_n)$ where (x_n) is a sequence in E_0 tending to x. The proof then consists in verifying that

(i) the sequence is Cauchy and so the limit exists

(ii) the limit does not depend on the choice of sequence

(iii) the resulting function f is uniformly continuous

(iv) f is Lipschitz if f_0 is (with $\operatorname{Lip} f = \operatorname{Lip} f_0$), and isometric if f_0 is.

The last part follows by letting $n \to \infty$ in

$$\rho\big(f_0(x_n), f_0(y_n)\big) \leq \big(\operatorname{Lip} f_0\big) d(x_n, y_n) \quad \big(\text{resp.} = d(x_n, y_n)\big),$$

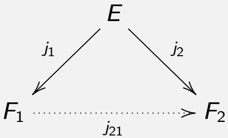
for suitable sequences (x_n) and (y_n); the rest is left as an exercise. □

Definition

A *completion* of a metric space E is a complete metric space F together with an isometry $j : E \to F$ which has dense range: $\overline{j(E)} = F$.

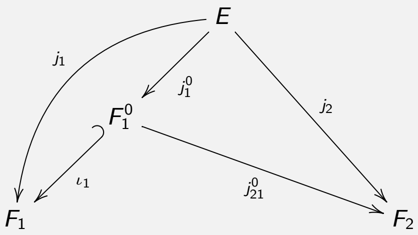
Theorem (Existence and uniqueness of metric space completions)

Let E be a metric space. Then E has a completion and if (F_1, j_1) and (F_2, j_2) are two completions then there is a unique surjective isometry $j_{21} : F_1 \to F_2$ satisfying $j_{21} \circ j_1 = j_2$:



Existence:
Recall that there is an isometry $\tilde{\jmath} : E \to X$, for a Banach space X. Setting $F = \overline{\tilde{\jmath}(E)}$ and letting $j : E \to F$ be the corestriction of $\tilde{\jmath}$, (F, j) is clearly a completion of E.

Uniqueness:



Proof.

Suppose that (F_1, j_1) and (F_2, j_2) are completions of E. Let j_1^0 be the corestriction of j_1 to $F_1^0 := \operatorname{Ran} j_1$, and let ι_1 be the inclusion map $F_1^0 \to F_1$. Thus j_1^0 is a surjective isometry satisfying $j_1 = \iota_1 \circ j_1^0$ (see the commutative diagram above). Since $j_{21}^0 := j_2 \circ (j_1^0)^{-1}$ is an isometry from the dense subset F_1^0 of F_1 into the complete space F_2, it has a unique extension to an isometry from F_1 to F_2. in other words there is a unique isometry $j_{21} : F_1 \to F_2$ satisfying $j_{21} \circ \iota_1 = j_2 \circ (j_1^0)^{-1}$, equivalently $j_{21} \circ \iota_1 \circ j_1^0 = j_2$, or $j_{21} \circ j_1 = j_2$.

To see that j_{21} is surjective, note that $\operatorname{Ran} j_{21}$ is dense in F_2 (since $j_{21}(F_1^0) = j_2(E)$) and complete (since the space F_1 is complete and the map j_{21} is isometric) and so equals F_2.

□

Theorem (Characterisation of compact metrisable spaces)

Let (S, τ) be a metrisable topological space. Then TFAE:

(i) (S, τ) *is compact;*

(ii) (S, d) *is complete for every metric d which metrises τ.*

Proof.

Exercise. [HINT: Show that if (x_n) is a sequence in S satisfying $\delta := \inf_{n \neq m} d(x_n, x_m) > 0$ for some metric for (S, τ) then, for a suitable sequence (α_n) in $\mathbb{R}_{>0}$,

$$\widetilde{d}(x, y) := \min \left(d(x, y), \inf_{n,m} \left\{ d(x, x_m) + |\alpha_n - \alpha_m|\delta + d(x_n, y) \right\} \right)$$

defines an equivalent incomplete metric.] □

Remark

Completion with respect to a totally bounded metric gives a compact space. In this connection see Urysohn's Metrisation Theorem below.

Definition

A topological space is *completely metrisable* if it can be metrised by a complete metric. Conversely, a metric space (E, d) is said to be *topologically complete* if there is an equivalent metric $\widetilde{d}$ on E such that $(E, \widetilde{d})$ is complete.

A countable intersection of open sets in a topological space is called a G_δ (set).

Example

The incomplete metric space $E =]-1, 1[$ has equivalent complete metric given by

$$d(s,t) = \left|f(t) - f(s)\right|, \text{ where } f : E \to \mathbb{R}, \quad t \mapsto \tan(\pi t/2).$$

Theorem (Characterisation of topological completeness)

For a metric space (E, d) TFAE:

(i) (E, d) *is topologically complete;*

(ii) E *is a* G_δ *in its completion.*

Example

The set of irrationals, in the relative topology from $\mathbb{R}$, is completely metrisable.

MS-5

- Polish spaces
- Prohorov-Lévy metrics, tightness and Prohorov's Theorem
- Polish path spaces

Definition
A topological space (E, τ) is *Polish* if it is separable and completely metrisable.

Theorem (Stability of the Polish property under countable products)
A countable product of Polish spaces is Polish.

Proof.
Combine two earlier results. □

Example (Prohorov metric)

Let $\widetilde{E}$ denote the set of Borel probability measures on a Polish space E. The *weak topology* for $\widetilde{E}$, which captures *weak convergence of probability measures*, has as subbase the family of sets

$$\left\{\nu \in \widetilde{E} : \left|\int f\, d\nu - \int f\, d\mu\right| < \varepsilon\right\} \qquad (\mu \in \widetilde{E},\ f \in C_{\mathrm{b}}(E;\mathbb{R}) \text{ and } \varepsilon > 0).$$

Let d be a complete metric for E *which is bounded by one* and, for $\mu, \nu \in \widetilde{E}$, setting

$$\widetilde{d}(\mu,\nu) := \inf\left\{\delta > 0 : \forall_{F \subset E, F \text{ closed}}\ \mu(F) < \nu(F^{(\delta)}) + \delta \text{ and } \nu(F) < \mu(F^{(\delta)}) + \delta\right\}.$$

Then,

- $\widetilde{d}$ defines a complete metric for $\widetilde{E}$ which metrises the weak topology;
- $\widetilde{d}$ extends d in the sense that $\widetilde{d}(\delta_x, \delta_y) = d(x,y)$ for $x, y \in E$ (here δ_x denotes the point measure at x of unit mass);
- $\mathrm{Conv}\,\{\delta_x : x \in E\}$ is dense in $\widetilde{E}$.

It follows that $\widetilde{E}$ inherits separability from E, and is therefore Polish too. Moreover (cf. Arzela–Ascoli), for $A \subset \widetilde{E}$, A is pre-compact iff A is tight:

$$\forall_{\varepsilon > 0} \exists_{K \subset E,\ \text{compact}} \forall_{\mu \in A} :\ \mu(K) > 1 - \varepsilon.$$

Example (More preservation of Polishness)

Let E be a Polish space, let d be a complete metrising metric and let $C = C(\mathbb{R}_+; E)$. Then the topology of uniform convergence on compact subintervals of $\mathbb{R}_+$ with respect to d is actually independent of the choice of d. Letting $\alpha : [0, \infty[\to [0, \infty[$ be the map $t \mapsto t/(1+t)$, (exercise) C is metrised by the complete metric given by

$$\rho(f, g) := \sum_{n \geq 0} 2^{-n} (\alpha \circ d_{\text{sup}}) \big(f|_{[0,n]}, g|_{[0,n]} \big).$$

Next let $\widetilde{\rho}$ be the corresponding metric on $\widetilde{C} := C(\mathbb{R}_+; \widetilde{E})$, where $\widetilde{E}$ is the Polish space of Borel probability measures on E in the topology of weak convergence. Fix a countable dense set D in $\widetilde{E}$. For $n \in \mathbb{N}$ let P_n denote the collection of functions $F : \mathbb{R}_+ \to \widetilde{E}$ obtained by piecewise-linear interpolation from choices

$$F(k2^{-n}) \in D \text{ for } k = 0, \ldots, 2^{2n}$$

and the stipulation $F(t) = F(2^n)$ for $t \geq 2^n$. (Note that F is indeed $\widetilde{E}$-valued!) Then the countable set $\bigcup_{n \in \mathbb{N}} P_n$ is dense in $\widetilde{C}$, so $\widetilde{C}$ is separable. To see that C is also separable, and so Polish, use the fact that it is homeomorphic to the topological subspace $\{\delta_{f(\cdot)} : f \in C\}$ of $\widetilde{C}$ (Exercise: supply the details.)

MS-6

- Nowhere density, meagre sets
- Genericity of nowhere-differentiability
- Baire Category Theorem
- Uncountability of Hamel bases
- Uniform Boundedness Principle

Definition

Let $A \subset S$ or a topological space S. Then A is *nowhere dense* if its closure has empty interior:

$$\operatorname{Int} \overline{A} = \emptyset;$$

A is *meagre* (or *of first category*) if it is expressible as a countable union of nowhere dense sets.

Examples

- Finite subsets of any topological space S in which singleton sets are closed but not open (e.g. nontrivial normed linear spaces). *Note*.
- Proper closed subspaces of a NLS are nowhere dense.
- In $\mathbb{R}^d$, $\mathbb{Q}^d$ is meagre but not nowhere dense; $\mathbb{Z}^d$ is nowhere dense.

TFAE (exercise)

- A is nowhere dense;
- $\overline{A}^c$ is dense;
- every nonempty open set U in S contains a nonempty open set V disjoint from A.

Example (Topological genericity of nowhere-differentiability)

Let $C = C_{\mathbb{R}}\big([0,1]\big)$ and let D be the set of functions in C which are right-differentiable *at some point in* $[0,1[$. Then D is a meagre subset of C.

Proof.

For each $n \in \mathbb{N}$, let A_n denote the set of functions $f \in C$ satisfying

$$\exists_{x\in[0,1-1/n]} \forall_{h\in]0,1-x[} \quad h^{-1}\big|f(x+h) - f(x)\big| \leq n.$$

Thus $D \subset \bigcup A_n$. Using the (sequential) compactness of $[0, 1-1/n]$ it is not hard to see that A_n is closed. Now let $n \in \mathbb{N}$, $f \in A_n$ and $\varepsilon > 0$. For $N \in \mathbb{N}$ let $h_N \in C$ be the sawtooth function which is zero at $\{k2^{-N} : k = 0, \ldots, 2^N\}$, one at $\big\{(2k-1)2^{-(N+1)} : k = 1, \ldots, 2^N\big\}$ and piecewise-linear in between. Then $f + \varepsilon h_N \in C \setminus A_n$ for sufficiently large N. It follows that A_n has no interior. Thus A_n is nowhere dense and the result follows. □

A randomly chosen continuous function is nowhere-differentiable

Remarks

- Nowhere differentiability for continuous functions is also generic, from a measure-theoretic point of view: the probability of a continuous function being differentiable is zero. (Probability being calculated according to the standard *Wiener measure* on C).
- There are many analogies between the topological notion of 'smallness' captured by the meagre property and measure-theoretic notion of null set. [For an interesting account of these — and the differences — see J.C. Oxtoby, Measure and Category (Springer, 1980).]

Theorem (Baire Category Theorem)

Any meagre set in a completely metrisable space S has dense complement.

Proof.

Let $A = \bigcup_{n\in\mathbb{N}} A_n$ for a sequence (A_n) of closed sets having empty interior, and let U be a nonempty open set in S. The result is proved by showing that $U \cap (S \setminus A) \neq \emptyset$. Let d be a complete metric for S. Nowhere density of each A_n means that we may choose recursively a sequence of open balls (G_n) starting with an open ball G_1 inside $U \cap (S \setminus A_1)$, such that

$$G_{n+1} \subset \operatorname{Int} F_n \cap (S \setminus A_n) \quad \text{for all } n \in \mathbb{N},$$

where F_n is the closed ball concentric with G_n of half its radius. Then

$$\bigcap_{n\in\mathbb{N}} F_n \subset \bigcap_{n\in\mathbb{N}} G_n \subset U \cap \bigcap_{n\in\mathbb{N}} (S \setminus A_n) = U \cap (S \setminus A),$$

so the result follows by Cantor's Intersection Theorem. □

Corollary

In a completely metrisable space S, every countable intersection of open dense sets (or dense G_δ's) is dense.

Infinite dimensional Banach spaces have uncountable Hamel bases

Corollary

Completely metrisable spaces are not meagre.

Corollary

No Banach space has a countably infinite Hamel basis.

Proof.

Suppose that $(e_n)_{n\in\mathbb{N}}$ is an indexed Hamel basis for a NLS X. Set

$$X_N = \mathrm{Lin}\left\{e_1, \ldots, e_N\right\} \text{ for } N \in \mathbb{N}.$$

Then X_N is closed (see below) and is a proper subspace of X, so X_N is nowhere dense. However $\bigcup_{N\in\mathbb{N}} X_N = X$, so the Baire Category Theorem implies that X cannot be complete. □

Theorem (Uniform Boundedness Principle)

Let $\mathcal{F} \subset C(S; E)$ for a completely metrisable space S and metric space E. If $\mathcal{F}$ is pointwise bounded then $\mathcal{F}$ is uniformly bounded on some nonempty open subset of S.

Proof.

Fix a point $e \in E$. By pointwise boundedness,

$$S = \bigcup_{n \in \mathbb{N}} S_n \quad \text{where} \quad S_n = \bigcap_{f \in \mathcal{F}} f^{-1}\big(B_n^E[e]\big).$$

Set $U_n = \mathrm{Int} S_n$ ($n \in \mathbb{N}$). Since each S_n is closed, the Baire Category Theorem implies that there is $M \in \mathbb{N}$ such that $U_M \neq \emptyset$. Thus

$$f(U_M) \subset B_M^E[e] \text{ for all } f \in \mathcal{F},$$

so $\mathcal{F}$ is uniformly bounded on the nonempty open set U_M. □

MS-7

- Lipschitz extensions
- Normality and Tietze' Extension Theorem
- Regularity and Urysohn's Metrisation Theorem
- Local compactness and one-point compactification
- Complete regularity and Stone-Cech compactification
- Stone-Weierstrass Theorem
- A tensor product of function spaces

Lipschitz functions have Lipschitz extensions

Proposition (Lipschitz extensions)

Let $f \in \mathrm{Lip}_{\mathbb{R}}(A)$ for a subset A of a metric space (E, d). Then there is a function $\widetilde{f} \in \mathrm{Lip}_{\mathbb{R}}(E)$ such that

$$\widetilde{f}\big|_A = f \quad \text{and} \quad \mathrm{Lip}\,\widetilde{f} = \mathrm{Lip}\,f.$$

Proof.

The set of Lipschitz extensions of f having the same Lipschitz constant L is partially ordered by graph-inclusion:

$$g \subset h \text{ means } \mathrm{Graph}\,g \subset \mathrm{Graph}\,h, \text{ or } \mathrm{Dom}\,g \subset \mathrm{Dom}\,h \text{ and } h|_{\mathrm{Dom}\,g} = g.$$

The resulting partially ordered set is easily seen to satisfy the hypotheses of Zorn's Lemma. It therefore suffices to show that f may be so extended by a single point $x \in E \setminus A$. This follows from the observation that, by the triangle inequality,

$$\big[f(a_1) + Ld(x, a_1)\big] - \big[f(a_2) - Ld(x, a_2)\big] \geq \big[f(a_1) - f(a_2)\big] + Ld(a_1, a_2) \geq 0$$

for all $a_1, a_2 \in A$, so that we may set $\widetilde{f}(x) := \inf_{a \in A} \big[f(a) + Ld(x, a)\big]$ to get the required inequality

$$-Ld(x, a) \leq \widetilde{f}(x) - f(a) \leq Ld(x, a) \quad \text{for all } a \in A.$$

Remarks

If f is also bounded then $\widetilde{f}$ may be chosen to be bounded with $\|\widetilde{f}\|_{\mathrm{sup}} = \|f\|_{\mathrm{sup}}$. To see this note that replacing $\widetilde{f}$ by $\alpha \circ \widetilde{f}$ where $\alpha(t) := \min\{\|f\|_{\mathrm{sup}}, t\}$ gives a bounded and Lipchitz extension of f with the same Lipchitz constant and sup-norm.

If $\mathbb{R}$ is replaced by $\mathbb{C}$ then $\mathrm{Lip}\,\widetilde{f} = \sqrt{2}\,\mathrm{Lip}\,f$ is the best that can be done.

Definition (Normal space)

A topological space is *normal* if for every pair of disjoint closed sets F_1 and F_2 there are disjoint open sets U_1 and U_2 such that $U_1 \supset F_1$ and $U_2 \supset F_2$.

Remarks

- A normal space is thus Hausdorff (also called T_2), provided that it is T_1 (that is its singleton sets are closed).
- (a) Pseudo-metrisable spaces are normal.
- (b) Compact Hausdorff spaces are normal.

Proof.

Case (a): Let $f(x) = d_{F_1}(x) - d_{F_2}(x)$ where (as usual) $d_A(x) := \operatorname{dist}(x, A)$, and set

$$U_1 := f^{-1}(\mathbb{R}_{<0}) \text{ and } U_2 := f^{-1}(\mathbb{R}_{>0}).$$

Case (b) is left as an exercise. □

Example (Function without continuous extension)

The bounded continuous function $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $t \mapsto \sin(1/t)$ has no extension $\widetilde{f} \in C_{\mathbb{R}}(\mathbb{R})$.

Theorem (Urysohn's Separation Lemma)

Let E and F be disjoint closed sets in a normal topological space S then there is $f \in C\big(S; [0,1]\big)$ such that $f \equiv 0$ on E and $f \equiv 1$ on F.

Theorem (Tietze Extension Theorem)

Let $f \in C_{\mathrm{b}}(F; \mathbb{R})$ for a closed set F in a normal topological space S. Then there is a function $\widetilde{f} \in C_{\mathrm{b}}(S; \mathbb{R})$ satisfying

$$\widetilde{f}\big|_F = f \quad \textit{and} \quad \|\widetilde{f}\|_{\sup} = \|f\|_{\sup}.$$

Remark

The theorem is also valid if $\mathbb{R}$ is replaced by $\mathbb{C}$ since, letting $\widetilde{f_1}$ and $\widetilde{f_2}$ be Tietze extensions of $\operatorname{Re} f$ and $\operatorname{Im} f$ respectively, $\gamma \circ \widetilde{f}$ does the job if $\widetilde{f} := \widetilde{f_1} + i\widetilde{f_2}$ and γ is the function defined by $\gamma(z) = \|f\|_{\sup}|z|^{-1}z$ if $|z| \geq \|f\|_{\sup}$ and $\gamma(z) = z$ otherwise.

Definition

A topological space S is *regular* if for every closed set F and element $p \notin F$ there are disjoint opens sets U and V such that $U \supset F$ and $p \in V$.

Remarks

- Equivalent condition: for each $p \in S$, every neighbourhood of p contains a closed neighbourhood of p, in other words the closed neighbourhoods of p form a neighbourhood base at p.
- A normal T_1 space is both regular and Hausdorff.

Theorem (Urysohn's Metrisation Theorem)

For a topological space S, TFAE

(i) *S is separable and pseudo-metrisable;*

(ii) *S is totally boundedly pseudo-metrisable;*

(iii) *S is second countable and regular.*

Definition
A topological space (S, τ) is *locally compact* if, for all $p \in S$, the compact neighbourhoods of p form a neighbourhood base at p.

Example
$\mathbb{K}^n$ is locally compact, for each $n \in \mathbb{N}$.

Proposition
For a Hausdorff topological space (S, τ) TFAE

(i) (S, τ) is locally compact;

(ii) every element $p \in S$ has a compact neighbourhood.

Proof.
Clearly (i) implies (ii). For the converse let H be an open neighbourhood of an element $p \in S$. For any compact neighbourhood K of p, since K is (compact and Hausdorff and so) regular in the relative topology, there is $U, V \in \tau$ such that $p \in U$, $H^c \cap K \subset V$ and $U \cap V \cap K = \emptyset$. In particular,

$$p \in U \cap K \subset V^c \cap K \subset H$$

so the compact set $V^c \cap K$ is a neighbourhood of p contained in H. $\square$

Remarks

- Compact spaces are locally compact.
- Locally compact Hausdorff spaces are regular.
- Finite products of locally compact spaces are locally compact.
- A NLS X is locally compact if and only if $\dim X < \infty$ (Riesz).
- Let $A \subset S$ be either open or closed. If S is locally compact then so is A, in its relative topology.
- The Baire Category Theorem holds if 'completely metrisable' is replaced by 'locally compact Hausdorff' in the hypotheses.

Proof.

Riesz' result will be proved later; the rest are left as an extended exercise.
[HINT: The proof given for the Baire Category Theorem adapts nicely.] □

Definition

A *compactification* of a topological space (S, τ) is a compact topological space $(\widetilde{S}, \widetilde{\tau})$ together with a topological embedding $j : S \to \widetilde{S}$ with dense range. A *topological embedding* is a map whose corestriction to its range is a homeomorphism.

Theorem (Alexandroff)

Every noncompact space (S, τ) has a one-point compactification $(\widetilde{S}, \widetilde{\tau})$ which is Hausdorff if and only if (S, τ) is locally compact and Hausdorff.

Proof.

Set $\widetilde{S} = S \cup \{\omega\}$ where ω denotes a point not in S and set $\widetilde{\tau} = \tau \cup \tau'$ where

$$\tau' := \big\{\{w\} \cup U : U \in \tau \text{ and } S \setminus U \text{ is compact}\big\}.$$

Exercise: Complete the proof. □

Remark

Note that a noncompact space is open in its one-point compactification.

Remarks

- The one-point compactification of S is often denoted $S \cup \{\infty\}$, with ∞ referred to as the *point at infinity*.
- Example: The one-point compactification of the complex plane is the Riemann sphere, with $j : \mathbb{C} \to S^2$ given by

  $$z \mapsto \left(a^{-1}\operatorname{Re} z,\ a^{-1}\operatorname{Im} z,\ (2a)^{-1}(|z|^2 - 1)\right) \text{ where } a = \left(|z|^2 + 1\right)/2.$$

  and inverse given by stereographic projection with North pole corresponding to the point at infinity.
- Example: The one-point compactification of $\mathbb{N}$ is $\{0\} \cup \{1/n : n \in \mathbb{N}\}$ with relative topology induced by that of $\mathbb{R}$, and map j given by $j(n) = 1/n$.

Remark

Since non-Hausdorff compactifications (of a Hausdorff space) would appear to be useless, some authors include the Hausdorff condition in the definition of compactification.

Definition

Let $f \in C(S;X)$ for a topological space S and NLS X. The *topological support* of f is the closure of its support: $\overline{f^{-1}\big(X \setminus \{0\}\big)}$. We say that

- f *has compact support* if $\overline{\operatorname{supp}} f$ is compact,
- f *vanishes at infinity* if, for all $\varepsilon > 0$, the following set is compact

$$f^{-1}\big(X \setminus B_\varepsilon^X(x)\big) = \big\{s \in S : \ \|f(s)\| > \epsilon\big\}.$$

Remarks

- For the corresponding classes of functions we have the subspace inclusions, the first of which is a dense inclusion (exercise):

$$C_\kappa(S;X) \subset C_0(S;X) \subset C_b(S;X),$$

  all being equal to $C(S;X)$ when S is compact.
- For noncompact S, there is a natural isometric identification

$$C_0(S;X) = \big\{f \in C(S \cup \{\infty\};X) : f(\infty) = 0\big\}.$$

Proposition ($C_\kappa(S)$ separates compacts from closed sets when S is a locally compact Hausdorff space)

Let $K \subset U \subset S$ for a compact set K and open set U of a locally compact Hausdorff space S. Then there is $f \in C_\kappa(S)$ such that $f \equiv 1$ on K and $\overline{\operatorname{supp}} f \subset U$.

Proof.

By normality of the compact Hausdorff space $S \cup \{\infty\}$, the point at infinity has a closed neighbourhood H_1 disjoint from K. Let $H = H_1 \cup U^c$. Being compact in S, K is closed in $S \cup \{\infty\}$. Therefore, by Urysohn's Lemma there is $f_1 \in C\big(S \cup \{\infty\}; [0,1]\big)$ such that $f \equiv 1$ on K and $f_1 \equiv 0$ on H. Setting $\widetilde{f} = (2f_1 - 1)_+$, we have $\widetilde{f} \equiv 1$ on K and

$$\overline{\operatorname{supp}}\, \widetilde{f} \subset J \subset H^c \subset U \text{ where } J := \big\{x \in S \cup \{\infty\} :\ f_1(x) \geq 1/3\big\}.$$

Since J is compact and (like U) excludes ∞, if follows that the function $f := \widetilde{f}|_S$ does the job. □

Definition

A topological space S is *completely regular* if, for any closed set F and element $p \notin F$ there is $f \in C(S)$ such that $f(p) = 1$ and f vanishes on F. A Hausdorff completely regular space is called a *Tychonoff space* (or $T_{3\frac{1}{2}}$-space).

Remarks

- f may be replaced by $\alpha \circ f$ where $\alpha : \mathbb{C} \to [0, 1]$ is the continuous function $z \mapsto \max\{0, \min\{1, \operatorname{Re} z\}\}$ and so $C(S)$ may be replaced by $C(S; [0, 1])$.
- Thus S is completely regular if $C(S)$ or $C(S; [0, 1])$ separates disjoint closed sets from elements of their complement.
- Any subset of a completely regular space is completely regular in the relative topology.
- Arbitrary products of completely regular spaces are completely regular.
- Hausdorff normal spaces (called T_4-spaces) are completely regular (by Urysohn's Separation Lemma).
- A locally compact Hausdorff space (need not be normal but is a topological subspace of its one-point compactification which is normal and so) is completely regular.

More remarks

Remarks

- For a completely regular space S, the initial topology of the family $C_b(S)$ coincides with the given topology of S (exercise).
- Any subset of a compact Hausdorff space is completely regular in its relative topology. The remarkable converse is established next.
- Recall the problem of extending the continuous function

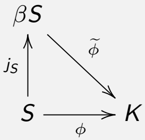
  $$]0,1] \to \mathbb{R},\ t \mapsto \sin(1/t)$$

  to its one-point compactification $[0,1]$.

Stone-Cech Compactification

Theorem

Any completely regular Hausdorff topological space S has a Hausdorff compactification $(\beta S, j_S)$ enjoying the following universal property: for every compact Hausdorff space K and map $\phi \in C(S; K)$ there is a unique map $\widetilde{\phi} \in C(\beta S; K)$ such that $\widetilde{\phi} \circ j_S = \phi$, in other words, every continuous map from S into a compact Hausdorff space extends uniquely to a continuous map from βS into that space.



The above, so-called Stone-Cech compactification, is unique in the following sense. Suppose that (β_1, j_1) and (β_2, j_2) are two such Hausdorff compactifications. Then there is a unique homeomorphism j_{21} from β_1 to β_2 such that $j_{21} \circ j_1 = j_2$.

Stone-Cech compactification: proof of existence

Uniqueness. By universality there are unique maps $j_{21} \in C(\beta_1; \beta_2)$ such that $j_{21} \circ j_1 = j_2$ and $j_{12} \in C(\beta_2; \beta_1)$ such that $j_{12} \circ j_2 = j_1$. It suffices now to show that the continuous maps j_{12} and j_{21} are mutually inverse. Since $j_2 = j_{21} \circ j_1 = j_{21} \circ j_{12} \circ j_2$, it follows that continuous functions $j_{21} \circ j_{12}$ and id_{β_2} into the Hausdorff space β_2 agree on the set $\mathrm{Ran}\, j_2$ which is dense in β_2, and so they are equal by the identity theorem. By symmetry, $j_{12} \circ j_{21} = \mathrm{id}_{\beta_1}$ too.

Existence. Let πS be the compact Hausdorff space $\prod_{f \in C} \overline{\mathrm{Ran} f}$ where $C = C_{\mathrm{b}}(S)$, and let $\eta = \eta_S : S \to \pi S$ be the 'evaluation map' given by $\eta(x)_f = f(x)\,(x \in S, f \in C)$. Then, for $f \in C$, the coordinate projection $\pi_f : \pi S \to \overline{\mathrm{Ran} f}$, $\alpha \mapsto \alpha_f$ satisfies $\pi_f\big(\eta(x)\big) = \eta(x)_f = f(x)$ for all $x \in S$, so $\pi_f \circ \eta = f$. In particular η is continuous. Since S is Hausdorff and completely regular, C separates the points of S and so η is injective.

Suppose now that H is a neighbourhood of a point $p \in S$. By complete regularity there is $f \in C$ such that $f(p) = 1$ and f vanishes on H^{c}. Setting $V = \{\alpha \in \pi S : \alpha_f \neq 0\} = \pi_f^{-1}\big(\overline{\mathrm{Ran} f} \setminus \{0\}\big)$, V is open in πS and contains $\eta(p)$. Moreover, for $x \in S$, $\eta(x) \in V \Rightarrow f(x) \neq 0 \Rightarrow x \in H$, so $V \cap \eta(S) \subset \eta(H)$.

Thus $\eta(H)$ is a neighbourhood of $\eta(p)$ in $\eta(S)$ for the relative topology. It follows that the bijection $S \to \eta(S)$ induced by η is open with respect to the relative topology on $\eta(S)$ and thus a homeomorphism. Letting j_S be the corestriction of η to $\beta S := \overline{\eta(S)}$, $(\beta S, j_S)$ is a Hausdorff compactification of S.

Proof.

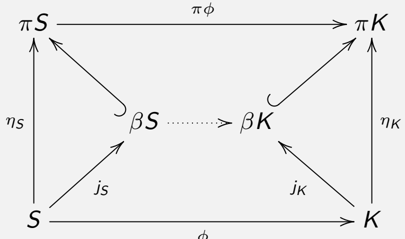
Let $\phi \in C(S;K)$ for a compact Hausdorff space K. Then (in the above notation) j_K is a homeomorphism (exercise) and the map

$$\pi\phi : \pi S \to \pi K \text{ given by } \pi\phi(\alpha)_g = \alpha_{g\circ\phi} \quad (g \in C(K))$$

satisfies $\pi\phi \circ \eta_S = \eta_K \circ \phi$ and $\pi_g \circ \pi\phi = \pi_{g\circ\phi}$ for all $g \in C(K)$ (exercise). It follows that $\pi\phi$ is continuous and maps βS into βK. Therefore, letting $\beta\phi$ denote the induced map $\beta S \to \beta K$, the function

$$\widetilde{\phi} := (j_K)^{-1} \circ \beta\phi$$

does the job; uniqueness follows from the Identity Theorem.



Theorem (Weierstrass)

Let $X = C_{\mathbb{K}}([a, b])$ *with supremum norm, and let* $\mathcal{P} = \mathcal{P}_{\mathbb{K}}([a, b])$ *denote the subspace of polynomial functions* $[a, b] \to \mathbb{K}$*. Then* $\mathcal{P}$ *is dense in* X*.*

Proof.

(i) Bernstein polynomials; (ii) Fourier series. □

Corollary

$\big(C_{\mathbb{K}}([a, b]), \|\cdot\|_{\sup}\big)$ *is separable.*

Proof.

Set

$$A = \Big\{ p \in \mathcal{P} : p(t) = \sum_{k=0}^{n} a_k t^k \text{ where } n \in \mathbb{Z}_+ \text{ and } a_k \in \mathbb{Q}\,(+i\mathbb{Q}) \text{ for } k = 0, 1, \ldots, n \Big\}$$

Then A is countable and is dense in $\mathcal{P}$, so

$$\overline{A} = \overline{\overline{A}} = \overline{\mathcal{P}} = X.$$

□

Lemma ($C_0(S)$ separates points when S is l.c. Hausdorff)

Let $x \neq y \in S$, for a locally compact Hausdorff space S. Then $f(x) \neq f(y)$ for some $f \in C_0(S)$.

Proof.

Exercise. $\big[$HINT: Apply the separating property proved earlier. If S is metrisable then one could take $f_x(z) = \gamma\big(d(x,z)\big)$ where $\gamma(t) = e^{-t^2}.\big]$. □

Recall:

our notation f^* for the pointwise complex conjugate of a $\mathbb{K}$-valued function f.

Theorem (Stone)

Let $\mathcal{F} \subset C_0(S)$ for a locally compact Hausdorff space S. If $\mathcal{F}$ separates the points of S and does not vanish identically at any point of S then $C_0(S)$ equals $\overline{{}^*\text{-}\mathrm{Alg}}\,\mathcal{F}$, the closed *-algebra generated by $\mathcal{F}$.

Corollary

Let $\mathcal{F} \subset C(K)$ for a compact Hausdorff space K. If $\mathcal{F}$ separates the points of K and contains a nonzero constant function then $\overline{{}^*\text{-}\mathrm{Alg}}\,\mathcal{F} = C(K)$.

Corollary

Let S and T be locally compact Hausdorff spaces. Then $S \times T$ is locally compact and Hausdorff, and

$$C_0(S)\underline{\otimes}C_0(T) := \mathrm{Lin}\left\{f \otimes g : f \in C_0(S), g \in C_0(T)\right\}$$

is dense in $C_0(S \times T)$.

Tychonoff $\equiv$ subspace of a product I^Λ

Theorem

For a topological space S, TFAE.

(i) *S is Tychonoff.*

(ii) *S topologically embeds into I^Λ for some set Λ.*

Here I denotes the closed unit interval $[0, 1]$.