# FUNCTIONAL ANALYSIS via MAGIC

# NORMED LINEAR SPACES (Topic 4)

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This section contains elements of the basic theory of normed linear spaces, Banach spaces and continuous linear maps between them. It includes a brief discussion of duality, bounded multilinear maps, differential calculus in Banach space and the weak$^*$-topology. In the Banach–Alaoglu Theorem, the noncompactness of unit balls is rectified — for dual Banach spaces — by switching topologies.

Two relevant facts proved in the appendix on topological vector spaces (TVS):
(1) finite-dimensional subspaces of a TVS are necessarily closed, and
(2) a continuous linear map from a dense subspace $X_0$ of a TVS $X$ into a complete TVS $Y$ uniquely extends to a continuous map from $X$ to $Y$, which is necessarily linear (CLE Theorem).

# NLS-1

- Operator norm
- Bounded $\equiv$ Continuous at 0 $\equiv$ Lipschitz
- Evaluations $\varepsilon : S \to L(\mathcal{F}; V)$ for $\mathcal{F} \subset \mathcal{F}(S; V)$
- $B(X; Y) \overset{\text{isom}}{\cong} C_b^L(B; Y)$, for $B = B_1^X[0]$

# Operator norm; boundedness for operators

## Definition

Let $X$ and $Y$ be NLS's. For $T \in L(X;Y)$, the *operator norm of* $T$ is

$$\|T\| := \sup_{\|n\| \leqslant 1} \|Tx\| \in [0, \infty]$$

and $T$ is said to be a *bounded operator* if $\|T\| < \infty$.

## Remarks

- Boundedness of an operator does *not* mean that $T$ is bounded as a function $X \to Y$ (which would imply that $T = 0$), however it *is* equivalent to boundedness of the function $T|_{B^X}$ where $B^X := B_1^X[0]$.
- With the convention that for $A \subset [0, \infty]$, $\sup A = 0$ and $\inf A = \infty$ if $A = \emptyset$, alternative expressions for $\|T\|$ abound; for example

$$\sup_{\|x\|=1} \|Tx\|, \quad \sup_{\|x\|<1} \|Tx\|, \quad \sup_{x \neq 0} \|Tx\|/\|x\|,$$
$$\inf\{\lambda > 0 : \forall_{x \in X} \|Tx\| \leqslant \lambda \|x\|\}.$$

- In view of the identity

$$\sup_{x \neq x'} \|Tx' - Tx\| \Big/ \|x' - x\| = \sup_{x \neq 0} \|Tx\| \Big/ \|x\|,$$

$T$ is bounded if and only if it is Lipschitz, in which case $\|T\| = \mathrm{Lip}\, T$.

# Bounded $\equiv$ Continuous at 0 $\equiv$ Lipschitz, for linear maps between NLS's

### Proposition

*Let $T \in L(X;Y)$ for NLS's $X$ and $Y$. Then TFAE:*

(i) *$T$ is continuous at 0;*

(ii) *$T$ is bounded;*

(iii) *$T$ is Lipschitz continuous.*

*When these hold $\|T\| = \operatorname{Lip} T$.*

### Proof.

If (i) holds then there is $\delta > 0$ such that $\|Tx\| = \|Tx - T0\| < 1$ for $x \in X$ satisfying $\|x\| = \|x - 0\| < \delta$. It follows that $\|Tx\| = \delta^{-1}\|T(\delta x)\| < \delta^{-1}$ for $\|x\| < 1$ and so $T$ is bounded. The rest follows from the remarks above. □

### Notation

Write $B(X;Y)$ for the collection of bounded $\equiv$ continuous $\equiv$ Lipschitz operators from $X$ to $Y$. It is an intersection of subspaces of the vector space $\mathcal{F}(X;Y)$:

$$B(X;Y) = L(X;Y) \cap C(X;Y).$$

$B(X;X)$ is abbreviated to $B(X)$.

# $\mathcal{F}^L(B;V)$ for a NLS closed unit ball $B$ and vector space $V$

Lemma

*For a NLS $X$ with closed unit ball $B$ and vector space $V$ over $\mathbb{K}$, set*

$$\mathcal{F}^L(B;V) := \left\{ f \in \mathcal{F}(B;V) : \ \forall_{(x,x',\lambda)\in\mathcal{T}} \ f(x+\lambda x') = f(x) + \lambda f(x') \right\}$$

*where* $\mathcal{T} := \left\{ (x,x',\lambda) \in B \times B \times \mathbb{K} : \ \ x + \lambda x' \in B \right\}$. *Then, for* $f \in \mathcal{F}(B;V)$, *TFAE:*

- $f \in \mathcal{F}^L(B;V)$;
- $\forall_{x,x'\in B} \ \forall_{t\in[0,1]} \ f((1-t)x + tx') = (1-t)f(x) + tf(x')$ *and* $\forall_{x\in B} \ \forall_{\lambda\in\mathbb{K}, \ |\lambda|=1} \ f(\lambda x) = \lambda f(x)$.

*Moreover,* $\mathcal{F}^L(B;V)$ *is a subspace of* $\mathcal{F}(B;V)$ *and the restriction map*

$$\rho : L(X;V) \to \mathcal{F}^L(B;V), \quad T \mapsto T|_B$$

*is a linear isomorphism.*

Proof.

Exercise. □

# Evaluations $\varepsilon : S \to L(\mathcal{F}; V)$, $\mathcal{F}$ subspace of $\mathcal{F}(S; V)$

## Definition

For any subspace $\mathcal{F}$ of a vector space of the form $\mathcal{F}(S; V)$, where $V$ is a vector space and $S$ a set, there is an associated map to *evaluations*:

$$\varepsilon = \varepsilon^{\mathcal{F}} : S \to L(\mathcal{F}; V), \quad s \mapsto \varepsilon_s \quad \text{where } \varepsilon_s(f) := f(s).$$

## Remarks

- Under the natural linear isomorphism

  $$\mathcal{F}(S; V) \to V^S = \prod_{s \in S} V,$$

  $\varepsilon_s^{\mathcal{F}}$ corresponds to $\pi_s|_{\mathcal{F}}$, a restriction of the coordinate projection.
- Linear bidual embedding: the case $S = U$, $\mathcal{F} = U^{\mathrm{dual}}$ and $V = \mathbb{K}$ for a vector space $U$ over $\mathbb{K}$,

  $$\varepsilon^{\mathcal{F}} = J_U : U \to L(U^{\mathrm{dual}}; \mathbb{K}) = (U^{\mathrm{dual}})^{\mathrm{dual}}.$$

- If $\mathcal{F} \subset C_b(S; X)$, for a topological space $S$ and NLS $X$, then each $\varepsilon_s^{\mathcal{F}}$ is *continuous* $\mathcal{F} \to X$ (with respect to the sup norm on $\mathcal{F}$).

# $C_b^L(B;Y)$ for a NLS $Y$ and closed unit ball $B$ of a NLS

**Proposition**

*For a NLS $Y$ and closed unit ball $B$ of a NLS $X$,*

$$C_b^L(B;Y) := C_b(B;Y) \cap \mathcal{F}^L(B;Y)$$

*is a closed subspace of the NLS $C_b(B;Y)$; moreover, $C_b^L(B;Y)$ is complete if $Y$ is a Banach space.*

**Proof.**

The first part follows from the third remark above and the identity

$$C_b^L(B;Y) = \bigcap_{(x,x',\lambda)\in\mathcal{T}} \mathrm{Ker}(\varepsilon_{x+\lambda x'} - \varepsilon_x - \lambda\varepsilon_{x'}),$$

for $\varepsilon = \varepsilon^{C_b(B;Y)}$, since $\mathrm{Ker}\, T = T^{-1}(\{0\})$ is closed for a bounded $\equiv$ continuous operator $T$ between NLS's. The second part follows from the fact that $C_b(B;Y)$ is complete if $Y$ is. □

# $B(X;Y) \overset{\text{isom}}{\cong} C_b^L(B;Y)$

## Theorem

*Let $X$ and $Y$ be NLS's and let $B = B^X$, the closed unit ball of $X$. Then:*

(a) *the operator norm* is *a norm on $B(X;Y)$;*

(b) *$T \mapsto T|_B$ defines an isometric isomorphism $B(X;Y) \to C_b^L(B;Y)$;*

(c) *$B(X;Y)$ is a Banach space if $Y$ is.*

## Proof.

Part (a) follows the earlier observations: $\operatorname{Lip} f > 0$ unless $f$ is constant, $\operatorname{Lip}(f+g) \leqslant \operatorname{Lip} f + \operatorname{Lip} g$ and $\operatorname{Lip}(\lambda f) = |\lambda| \operatorname{Lip} f$, coupled with the fact that there are no nonzero constant linear maps. Part (b) follows from the definitions, and part (c) follows from (b) and the previous proposition. □

## Remarks

- The topological dual $X^* := B(X;\mathbb{K})$ of any NLS $X$ is a Banach space.
- If $X$ is a Banach space then so is $B(X)$.
- The *operator norm inequality* relates the norms of $X$, $Y$ and $B(X;Y)$:

$$\|Tx\| \leqslant \|T\|\|x\| \quad \text{for } x \in X,\ T \in B(X;Y).$$

## NLS-2

- F. Riesz' dichotomy
- Banach–Steinhaus Theorem
- Open Mapping, Closed Graph and Banach Isomorphism

## Lemma (F. Riesz' Geometric Lemma)

*Let $Y$ be a proper closed subspace of a NLS $X$ and let $\varepsilon > 0$. Then there is a unit vector $x \in X$ satisfying*

$$\operatorname{dist}(x, Y) > 1 - \varepsilon.$$

## Proof.

Let $a \in X \setminus Y$ and suppose without loss of generality that $\varepsilon < 1$. Set

$$d := \operatorname{dist}(a, Y).$$

Since $Y$ is closed and $a \notin Y$, $d > 0$. Choose $a_0 \in Y$ such that $\|a - a_0\| < (1 - \varepsilon)^{-1} d$ and note that $\|a - a_0\| > 0$ since $a \notin Y$. Set $x = \|a - a_0\|^{-1}(a - a_0)$. Then, for all $y \in Y$,

$$\|x - y\| = \|a - a_0\|^{-1} \Big\| a - a_0 - \|a - a_0\| y \Big\| \geqslant \|a - a_0\|^{-1} d > 1 - \varepsilon.$$

□

## Remark

In Hilbert space there is no need for $\varepsilon$ – any unit vector in the nontrivial subspace $Y^{\perp}$ (orthogonal complement) will do.

### Theorem (F. Riesz)

*Let $B$ be the closed unit ball of a NLS $X$. Then TFAE:*

(i) *$B$ is compact;*

(ii) $\dim X < \infty$.

### Proof.

We have already seen that the Heine–Borel Theorem holds in finite-dimensional NLS's , so (ii) implies (i). Suppose therefore that $X$ is infinite-dimensional. Finite-dimensional subspaces of $X$ are closed; they are also obviously proper. Riesz' Geometric Lemma therefore ensures that, starting with an arbitrary unit vector $x_1$, there is a sequence of unit vectors $(x_n)$ in $X$ satisfying

$$\operatorname{dist}\big(x_{n+1}, \operatorname{Lin}\{x_1, \ldots, x_n\}\big) \geqslant \tfrac{1}{2} \quad \text{for all } n \in \mathbb{N}.$$

Since $\|x_p - x_q\| \geqslant \frac{1}{2}$ for $p \neq q$, this sequence can have no convergent subsequence. Therefore $B$ is not sequentially compact. Thus (i) implies (ii). □

### Remark

In Hilbert space the proof is easier: take an orthonormal sequence and apply Pythagoras. In TVS (i) is replaced by: $X$ is locally compact.

### Theorem

*Let $\mathcal{B} \subset B(X; Y)$ for a Banach space $X$ and NLS $Y$. If $\mathcal{B}$ is pointwise bounded then it is norm bounded:*

$$\sup_{T \in \mathcal{B}} \|T\| < \infty.$$

### Proof.

Exercise. [HINT: Apply the Uniform Boundedness Principle to the family of continuous functions $\mathcal{F} := \{T|_{B_1^X[0]} : T \in \mathcal{B}\}$ and use the homogeneity of NLS's.] □

### Remarks

- As with the Uniform Boundedness Principle, the target spaces for the elements of $\mathcal{B}$ can vary.
- The Banach–Steinhaus Theorem has a TVS generalisation in which $X$ is an F-space, the hypothesis is the same and the conclusion is: equicontinuity of the family $\mathcal{B}$.

An F-space is a TVS which is metrisable by a complete translation-invariant metric. Thus Banach spaces are F-spaces.

### Definition

A function between topological spaces is called *open* if it maps open sets to open sets.

### Remarks

- Thus a continuous bijection is a homeomorphism if and only if it is open.
- Coordinate projections from a product space are important examples of open maps (exercise).

Recall that the graph of any continuous function taking values in a Hausdorff space is closed.

### Lemma

*Let $f \in \mathcal{F}(S; T)$ be a continuous bijection, between topological spaces $S$ and $T$, with closed graph $G$. Then $f^{-1}$ has a closed graph too.*

### Proof.

This follows from the fact that $\mathrm{Graph}(f^{-1}) = \Pi G$ where $\Pi$ is the flip map $S \times T \to T \times S$, $(s, t) \mapsto (t, s)$ which is clearly a homeomorphism. □

# A functional-analytic triumvirate

Let $T \in L(X; Y)$ for F-spaces $X$ and $Y$.

Theorem (Open Mapping Theorem)

*If $T$ is continuous and surjective then it is open.*

Theorem (Closed Graph Theorem)

*If $T$ has closed graph then it is continuous.*

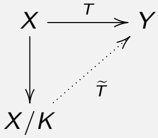
Theorem (Banach Isomorphism Theorem)

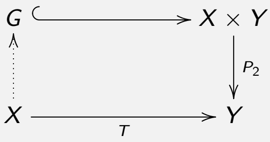
*If $T$ is continuous and bijective then it is a homeomorphism.*

In particular these results hold for Banach spaces $X$ and $Y$.

**Exercise**

Show that each of these three results may be deduced from any of the others. [HINT: For CG $\implies$ BI apply the lemma; for BI $\iff$ OM and BI $\implies$ CG, consider the commutative diagrams below.]



# NLS-3

- Topological dual spaces
- Topological bidual embedding; reflexivity
- NLS completion
- Topological transpose
- Norming a direct sum of NLS's
- $\ell^p$, $p \in [1, \infty]$; conjugate exponent
- $(\ell^p)^* \equiv \ell^{p'}$ for $p \in [1, \infty[$, $(c_0)^* \equiv \ell^1$
- Topological complements

### Definition

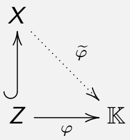
The *topological dual* of a TVS $X$ is the vector space $X^{\mathrm{dual}} \cap C(X : \mathbb{K}) = CL(X; \mathbb{K})$, denoted by $X^*$. When $X$ is a NLS, $X^*$ is itself a Banach space: $X^* = B(X; \mathbb{K})$.

### Theorem (Extension of continuous linear functionals)

*Let $\varphi \in Z^*$ for a subspace $Z$ of a Hausdorff locally convex TVS $X$. Then there is $\widetilde{\varphi} \in X^*$ extending $\varphi : \widetilde{\varphi}|_Z = \varphi$. If $X$ is a NLS then $\widetilde{\varphi}$ may be chosen so that $\|\widetilde{\varphi}\| = \|\varphi\|$.*

### Proof.

The NLS version is a direct consequence of the Hahn–Banach Extension Theorem. □



**Corollary ($X^*$ separates $X$)**

*Let $X$ be a Hausdorff locally convex TVS (e.g. a NLS). Then $X^*$ separates the elements of $X$.*

**Proof.**

Exercise. □

**Examples**

- $c_{00} \to \mathbb{C}$, $x \mapsto \sum x_i$ is a *discontinuous* linear functional.
- $L^p[0,1]^* = \{0\}$ for $p \in \,]0,1[$.

**Definition**

For a NLS $X$, the evaluation map

$$\eta = \eta^X : X \to X^{**}, \; x \mapsto \eta_x \quad \text{where } \eta_x(\varphi) := \varphi(x)$$

is called the *topological bidual embedding* of $X$.

**Proposition**

*For any NLS $X$, $\eta^X$ is isometric.*

**Proof.**

Exercise. [Hint: Clearly $\eta^X$ is a contraction. To see that $\eta_x \in X^{**}$ has norm 1, when $x$ is a unit vector, let $\widetilde{\varphi} \in X^*$ be an extension of the functional $\varphi : \mathbb{K}x \to \mathbb{K}$, $\lambda x \mapsto \lambda$, with the same norm, 1.] □

**Definition**

A *reflexive* NLS is a NLS $X$ for which $\eta^X$ is (surjective and thus) an isometric isomorphism.

**Warning**

There are nonreflexive spaces that are nevertheless isometric to their topological biduals.

# NLS completion

## Definition

A *completion* of a NLS $X$ is a Banach space $Z$ together with a linear isometry $J : X \to Z$ with dense range.

## Proposition

(a) *Every NLS has a completion.*

(b) *If $(Z_1, J_1)$ and $(Z_2, J_2)$ are completions of a NLS $X$ then there is a unique isometric isomorphism $J_{21} : Z_1 \to Z_1$ satisfying $J_{21} J_1 = J_2$.*

## Proof.

(a) Let $Z = \overline{\eta^X(X)}$ where $\eta^X : X \to X^{**}$ is the topological bidual embedding, and let $J$ be the induced map $X \to Z$. Since $\eta^X$ is isometric, $(Z, J)$ is a completion of $X$.

(b) On the one hand, by the uniqueness of metric space completions there is a unique surjective isometry $J : Z_1 \to Z_2$ satisfying $J \circ J_1 = J_2$. On the other hand, by the CLE Theorem, the isometry $J|_{J_1(X)}$ has a unique continuous extension to a map $Z_1 \to Z_2$ and this extension is linear. It follows that $J$ is an isometric isomorphism of Banach spaces.

□

**Lemma**

*Let $R \in B(X;Y)$ for NLS's $X$ and $Y$. Its linear transpose $R^\top \in L(Y^{\mathrm{dual}}; X^{\mathrm{dual}})$ restricts to a bounded operator $R^{\mathrm{top}\top} \in B(Y^*; X^*)$ called the* topological transpose *of $R$.*

**Proof.**

Straightforward. □

**Proposition**

*Let $R \in B(X;Y)$ for NLS's $X$ and $Y$. Then:*

(a) $\|R^{\mathrm{top}\top}\| = \|R\|$;

(b) $(R^{\mathrm{top}\top})^{\mathrm{top}\top} \circ \eta^X = \eta^Y \circ R$.

**Proof.**

Exercise. [HINT: It follows from the definition that $\|R^{\mathrm{top}\top}\| \leqslant \|R\|$; the reverse inequality may be extracted from (b) using the isometry of $\eta^X$ and $\eta^Y$.] □

**Remark**

The topological transpose of $R$ is also known as the *adjoint* of $R$ and is often denoted by $R^*$ (or $R'$); we steer clear of this terminology and notation to avoid confusion with Hilbert space adjoints.

For each $p \in [1, \infty]$, the topological direct sum of NLS's $X$ and $Y$ is normed by

$$\|(x, y)\| := \begin{cases} \left(\|x\|^p + \|y\|^p\right)^{1/p} & p \in [1, \infty[, \\ \max\left\{\|x\|, \|y\|\right\} & p = \infty. \end{cases}$$

We denote the resulting NLS by $X \oplus_p Y$.

**Proposition**

*For NLS's $X$ and $Y$ and $p \in [1, \infty]$, there is a natural isometric isomorphism*

$$(X \oplus_p Y)^* \cong X^* \oplus_{p'} Y^*,$$

*where $p' \in [1, \infty]$ is the exponent conjugate to $p$, namely $1 + (p-1)^{-1}$.*

**Proof.**

Exercise – find it and prove it. □

**Remark**

The choice of $p$ here is a matter of convenience. In case $X$ and $Y$ are Hilbert spaces the natural choice is $p = 2$ since $X \oplus_2 Y$ is then a Hilbert space.

For $p \in [1, \infty[$, the collection of sequences $z = (z_n)$ in $\mathbb{K}$ satisfying

$$\|z\|_p := \left( \sum_{n=1}^{\infty} |z_n|^p \right)^{1/p} < \infty$$

is a subspace of the vector space $\mathcal{F}_{\mathbb{K}}(\mathbb{N})$, on which $\|\cdot\|_p$ defines a norm, with respect to which the space is complete. The resulting Banach space is denoted by $\ell^p_{\mathbb{K}}$, or $\ell^p$ if $\mathbb{K} = \mathbb{C}$.

## Remark

- This is consistent with our definition of $\ell^p_{\mathbb{K}}$ and $\ell^p$ for $p = \infty$.
- That $\|\cdot\|_p$ satisfies the triangle inequality is a straightforward consequence of Minkowski's inequality

$$\left( \sum_{n=1}^{N} |z_n + w_n|^p \right)^{1/p} \leqslant \left( \sum_{n=1}^{N} |z_n|^p \right)^{1/p} + \left( \sum_{n=1}^{N} |w_n|^p \right)^{1/p}.$$

- The *conjugate exponent* of $p \in [1, \infty]$ is $p' := 1 + (p-1)^{-1} \in [1, \infty]$.
- If $z \in \ell^p_{\mathbb{K}}$ and $w \in \ell^{p'}_{\mathbb{K}}$ where $p \in [1, \infty]$, then $zw := (z_n w_n) \in \ell^1_{\mathbb{K}}$ and

$$\|zw\|_1 \leqslant \|z\|_p \|w\|_{p'} \quad \text{(Hölder's Inequality).}$$

# Classical dualities: $(c_{0,\mathbb{K}})^* \overset{\text{isom}}{\cong} \ell^1_{\mathbb{K}}$; $(\ell^p_{\mathbb{K}})^* \overset{\text{isom}}{\cong} \ell^{p'}_{\mathbb{K}}$ for $p \in [1,\infty[$

The collection of sequences in $\mathbb{K}$ which converge (respectively converges to zero) is denoted by $c_{\mathbb{K}}$ (respectively $c_{0,\mathbb{K}}$). These form closed subspaces of the Banach space $\ell^\infty_{\mathbb{K}}$. This notation is consistent with earlier notation – if $\mathbb{N}$ is endowed with the discrete topology then

$$\ell^\infty_{\mathbb{K}} = C_b(\mathbb{N};\mathbb{K}) \equiv C(\beta\mathbb{N};\mathbb{K}), \quad c_{\mathbb{K}} \equiv C(\mathbb{N}\cup\{\infty\};\mathbb{K}) \quad \text{and} \quad c_{0,\mathbb{K}} = C_0(\mathbb{N};\mathbb{K}).$$

## Proposition

*Let $p \in [1,\infty]$.*

(a) *For $a \in \ell^{p'}_{\mathbb{K}}$, $\varphi_a : z \mapsto \sum_{n=1}^\infty a_n z_n$ defines a bounded linear functional on $\ell^p_{\mathbb{K}}$; for $a \in \ell^1_{\mathbb{K}}$, the same formula defines a bounded linear functional $\psi_a$ on $c_{0,\mathbb{K}}$.*

(b) *The map $\Phi_{p'} : a \mapsto \varphi_a$ is isometric $\ell^{p'}_{\mathbb{K}} \to (\ell^p_{\mathbb{K}})^*$, and surjective thus an isometric isomorphism if $p \neq \infty$.*
*The map $\Psi : a \mapsto \psi_a$ is an isometric isomorphism $\ell^1_{\mathbb{K}} \to (c_{0,\mathbb{K}})^*$.*

- *$\ell^p_{\mathbb{K}}$ is reflexive if and only if $p \in ]1,\infty[$; $c_{0,\mathbb{K}}$ and $c_{\mathbb{K}}$ are not reflexive.*
- *$\ell^p_{\mathbb{K}}$ is separable if and only if $p \in [1,\infty[$; $c_{0,\mathbb{K}}$ and $c_{\mathbb{K}}$ are separable.*

Exercise. Prove this. [Hint: For $X = \ell^p_{\mathbb{K}}$ and $p \in [1,\infty[$, verify that $\eta^X = \left(\Phi_{p'}^{-1}\right)^{\text{top}\top} \circ \Phi_p$.]

### Definition

A closed subspace $Z$ of a TVS $X$ is *topologically complemented* if there is a closed subspace $Y$ of $X$ such that $X$ is the internal direct sum of $Y$ and $Z$:

$$X = Y + Z \text{ and } Y \cap Z = \{0\}.$$

### Remark

If $P \in L(X)$ is a continuous idempotent $(P^2 = P)$ then $\operatorname{Ran} P$ is topologically complemented by $\operatorname{Ker} P$. Note: $I - P$ is idempotent too and $\operatorname{Ker}(I - P) = \operatorname{Ran} P$. Idempotents in $L(X)$ are also called *projections*.

### Theorem*

*Let $X$ be a Banach space. Then TFAE:*

(i) *every closed subspace of $X$ is topologically complemented;*

(ii) *$X$ is linearly homeomorphic to a Hilbert space.*

# Topological complements for finite (co-)dimensional subspaces

## Proposition

*Let $Z$ be a closed subspace of a Hausdorff TVS $X$.*

(a) *Suppose that $Z$ has finite codimension. Then it is topologically complemented.*

(b) *Suppose that $Z$ has finite dimension and $X$ is locally convex. Then $Z$ is topologically complemented.*

## Proof.

(a) Let $(\widetilde{e}_i)_{i=1}^n$ be an ordered basis for $X/Z$ and choose

$$e_i \in Q^{-1}(\widetilde{e}_i) \quad \text{for } i = 1, \ldots, n,$$

where $Q$ denotes the quotient map $X \to X/Z$. Then $\mathrm{Lin}\{e_1, \ldots, e_n\}$ is complementary to $Z$ and, being finite-dimensional, is closed.

(b) Let $(e_i)_{i=1}^n$ be a basis for $Z$ and let $(\varphi_i)_{i=1}^n$ be its dual basis. By the Hahn–Banach Theorem, each $\varphi_i$ extends to a continuous linear functional $\widetilde{\varphi}_i$ on $X$. The map

$$X \to \mathbb{K}^n, \quad x \mapsto (\widetilde{\varphi}_1(x), \ldots, \widetilde{\varphi}_n(x))$$

is a continuous linear surjection; its kernel is a topological complement of $Z$. □

# NLS-4

- Weak*-topology
- Banach–Alaoglu Theorem
- NLS $X \hookrightarrow C(K;\mathbb{K})$, for $K$ compact Hausdorff

# Weak$^*$-topology on the dual of a NLS

### Definition

Let $Y = X^*$ for a NLS $X$. The *weak$^*$-topology* on $Y$ is the initial topology for the family

$$\{\widehat{x} := \eta^X(x) : x \in X\} \subset \mathcal{F}(Y; \mathbb{K}).$$

### Lemma

*Let $Y = X^*$ for a NLS $X$. The weak$^*$-topology on $Y$ coincides with the relative topology when $Y$ is viewed as a subspace of $\mathbb{K}^X$ (in the product topology).*

### Proof.

For $x \in X$ and $W \subset \mathbb{K}$ open, set

$$U_{x,W} := \{\varphi \in Y : \varphi(x) \in W\} = \widehat{x}^{-1}(W) \quad \text{and} \quad V_{x,W} := \{f \in \mathbb{K}^X : f(x) \in W\}.$$

Then the weak$^*$-topology on $Y$ is generated by $\{U_{x,W}\}$, and the product topology on $\mathbb{K}^X$ is generated by $\{V_{x,W}\}$. The result therefore follows since

$$U_{x,W} = V_{x,W} \cap Y.$$

□

### Corollary

*Weak$^*$-topologies are Hausdorff.*

### Theorem (Banach–Alaoglu Theorem)

*The closed unit ball of the dual space of a NLS is weak$^*$-compact.*

### Proof.

Let $K$ denote the closed unit ball of $X^*$, for a NLS $X$. By the lemma it suffices to show that $K$ is a compact subset of $\mathbb{K}^X$. This follows from Tychonoff's Theorem and the representation

$$K = \prod_{x \in X} B^{\mathbb{K}}_{\|x\|}[0] \;\cap\; L(X;\mathbb{K})$$

since

$$L(X;\mathbb{K}) = \bigcap_{x,x' \in X, \lambda \in \mathbb{K}} (\varepsilon_{x+\lambda x'} - \varepsilon_x - \lambda \varepsilon_{x'})^{-1}(\{0\})$$

and each of the evaluation functionals

$$\varepsilon_x : \mathcal{F}(X;\mathbb{K}) = \mathbb{K}^X \to \mathbb{K}, \quad f \mapsto f(x)$$

is continuous with respect to the product topology. □

# Every NLS embeds isometrically into a $C(K;\mathbb{K})$ for $K$ compact Hausdorff

**Theorem**

*Let $X$ be a NLS over $\mathbb{K}$. Then there is a linear isometry $J : X \to C(K;\mathbb{K})$, for a compact Hausdorff space $K$.*

**Proof.**

Let $K$ be the closed unit ball of $X^*$ in its weak$^*$-topology, and define a map

$$J : X \to C(K;\mathbb{K}) \quad \text{by } Jx = \widehat{x}|_K.$$

in terms of the bidual embedding $\eta^X : X \to X^{**}$, $x \mapsto \widehat{x}$. Then $J$ is linear and inherits isometry from $\eta^X$: for all $x \in X$,

$$\|Jx\| = \sup\left\{ \left|\widehat{x}(\varphi)\right| : \varphi \in K \right\} = \sup\left\{ \left|\widehat{x}(\varphi)\right| : \|\varphi\| \leqslant 1 \right\} = \|\widehat{x}\| = \|x\|$$

so $J$ is isometric. □

**Remark**

The following are equivalent:

- $X$ is separable;
- $K$ is metrisable;
- $C(K;\mathbb{K})$ is separable.

## NLS-5

- Bounded multilinear maps
- Norm for bounded multilinear maps
- Differential calculus in Banach space
- Chain Rule

**Definition**

Let $\alpha \in ML(X_1, \ldots, X_n; Z)$, for NLS's $X_1, \ldots, X_n, Z$. Then $\alpha$ is said to be *bounded* if there is $C \geqslant 0$ such that

$$\|\alpha(x_1, \ldots, x_n)\| \leqslant C\|x_1\| \cdots \|x_n\| \quad \text{for all } x_1 \in X_1, \ldots, x_n \in X_n.$$

**Proposition**

*Let $\alpha \in ML(X_1, \ldots, X_n; Z)$ for NLS's $X_1, \ldots, X_n, Z$. Then TFAE:*

(i) $\alpha$ *is bounded;*

(ii) $\alpha$ *is continuous.*

**Proof.**

Exercise. [HINT: We already covered the case $n = 1$, try $n = 2$.] □

### Properties

- The collection of bounded multilinear maps $X_1 \times \cdots \times X_n \to Z$ forms a subspace of the vector space $ML(X_1, \ldots, X_n; Z)$, denoted by $B(X_1, \ldots, X_n; Z)$, on which

$$\|\alpha\| := \sup_{\|x_1\|, \ldots, \|x_n\| \leqslant 1} \|\alpha(x_1, \ldots, x_n)\|$$

  defines a norm. Moreover, $B(X_1, \ldots, X_n; Z)$ is a Banach space if $Z$ is.
- There are natural isometric isomorphisms

$$B\big(X; B(Y; Z)\big) \overset{\text{isom}}{\cong} B(X, Y; Z) \overset{\text{isom}}{\cong} B(Y, X; Z).$$

  In particular, $B(X; Y^*) \overset{\text{isom}}{\cong} B(X, Y; \mathbb{K})$.

### Definition

Let $f : U \subset X \to Y$, for Banach spaces $X$ and $Y$ and an open subset $U$ of $X$. Then $f$ is (*Fréchet*) *differentiable at* $a \in U$ if there is $T \in B(X;Y)$ such that

$$\|h\|^{-1}\big\|f(a+h) - f(a) - Th\big\| \to 0 \quad \text{as } h \to 0.$$

### Remarks

- If such a $T$ exists then it is unique (exercise) and is denoted by $f'(a)$.
- $f$ is said to be *differentiable* if it is differentiable at all points $a$ of its domain $U$.
- If $f$ is differentiable then $f' : U \subset X \to B(X;Y)$, and so one can speak of twice, $\ldots$, $n$ times or infinite differentiability.
- Let $f : X \to Y$ be of the form $x \mapsto c + Tx$ $(c \in Y,\ T \in B(X;Y))$. Then $f$ is differentiable and $f'(a) = T$ for all $a \in X$.
- Let $\alpha \in B(X, Y; Z)$. Then $\alpha$, viewed as a map $X \oplus Y \to Z$, is differentiable and $\alpha'(a,b) \in B(X \oplus Y; Z)$ is given by $\alpha'(a,b)(x,y) = \alpha(a,y) + \alpha(x,b)$.

**Remark**

A useful way of thinking about differentiability at $a \in U$ is as follows: there is $T \in B(X;Y)$ and, for sufficiently small $r > 0$, a function $\varepsilon : B_r^X(0) \to Y$, such that

$$f(a+h) = f(a) + Th + \|h\|\varepsilon(h) \quad \big(\text{for } \|h\| < r\big) \text{ where } \varepsilon(h) \to 0 \text{ as } h \to 0.$$

**Proposition (Chain Rule)**

*Let $f : U \subset X \to Y$ be differentiable at $a$, suppose that $f(a) \in V$ and let $g : V \subset Y \to Z$ be differentiable at $f(a)$. Then $g \circ f$ is differentiable at $a$ and*

$$(g \circ f)'(a) = g'\big(f(a)\big)f'(a).$$

**Proof.**

Exercise. □

# Examples/exercises

- Let $f : GL_n(\mathbb{C}) \subset M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be the *inversion map* $A \mapsto A^{-1}$. Then $GL_n(\mathbb{C})$ is open in $M_n(\mathbb{C})$ (see Banach Algebras Section) and

$$f'(A)B = -A^{-1}BA^{-1}.$$

- Let $f : GL_n(\mathbb{C}) \subset M_n(\mathbb{C}) \to \mathbb{C}$ be the *determinant* map $A \mapsto \det A$. Then $f$ is differentiable – what is its derivative? [HINT: The answer is expressible in terms of the Trace.]