FUNCTIONAL ANALYSIS
via MAGIC

TOPOLOGICAL VECTOR SPACES (Appendix 1)

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In this section we see how topology and linear algebra combine at the most fundamental level, through the concept *vector topology*. The wider context (than Banach spaces and Hilbert spaces) affords essential flexibility in the uses of functional analysis, for example in its applications to PDE's; aspects of it are also needed for the study of Banach/Hilbert spaces and their operators – their weak, weak* and weak operator topologies being crucial tools.

Finite-dimensional vector spaces have a unique Hausdorff vector topology; moreover, for a Hausdorff TVS, finite-dimensionality is equivalent to local compactness (F. Riesz' dichotomy). The quotient operation works well for *closed* subspaces, however adding closed subspaces is delicate.

Also included is a summary of the main aspects of *uniform spaces* – the natural territory for discussing uniform continuity and completeness. Uniformisability of a topology is equivalent to its complete regularity, providing a link with Topic 3. On the other hand, TVS's have a unique translation-invariant uniformity, providing the link with the main business of this section – and also applications to completion and continuous linear extension.

Locally convex TVS's form a most important class, for example in applications to distribution theory (generalised functions such as the Dirac delta). Normability of a TVS is characterised in a celebrated theorem of Kolmogorov. Other echoes of results from Topic 3 include first-countability as a characterisation of (pseudo-)metrisability for a TVS, which then has a translation-invariant (pseudo-)metrisation whose balls satisfy a useful symmetry condition (they are 'balanced').

TVS-1

- Translation, multiplication, dilation and –kets
- Balls in a NLS as translates of dilates of the unit ball
- Absorbing, balanced and convex sets
- Minkowski functionals and seminorms

For a vector space V over $\mathbb{K}$ the following operations are fundamental:

$$T_a : V \to V,\ v \mapsto v{+}a; \quad D_\mu : V \to V,\ v \mapsto \mu v \quad \text{and} \quad |c\rangle : \mathbb{K} \to V,\ \lambda \mapsto \lambda c.$$

$(a \in V,\ \mu \in \mathbb{K} \setminus \{0\},\ c \in V \setminus \{0\}.)$ Each *translation* T_a is an affine isomorphism (preserving convex combinations), each *multiplication* D_μ (or *dilation* if $\mu \in \mathbb{R}_{>0}$) is a linear isomorphism, and each nonzero *-ket* $|c\rangle$ is a linear injection. They satisfy

$$T_0 = D_1 = \mathrm{id}_V, \quad T_a \circ T_b = T_{a+b}, \quad D_\mu D_\nu = D_{\mu\nu},$$
$$D_\mu \circ T_a = T_{\mu a} \circ D_\mu \quad \text{and} \quad D_\mu |c\rangle = |\mu c\rangle.$$

Even more fundamental, of course, are *summation* and *scalar multiplication* themselves:

$$s : V \times V \to V,\ (v, v') \mapsto v + v' \quad \text{and} \quad m : \mathbb{K} \times V \to V,\ (\lambda, v) \mapsto \lambda v.$$

Note that they are related by

$$T_a = s \circ f_a, \quad D_\mu = m \circ g_\mu \quad \text{and} \quad |c\rangle = m \circ h_c$$

for the (affine linear, injective) maps

$$f_a : V \to V \oplus V,\ v \mapsto (v, a), \quad g_\mu : V \to \mathbb{K} \oplus V,\ v \mapsto (\mu, v) \quad \text{and}$$
$$h_c : \mathbb{K} \to \mathbb{K} \oplus V,\ \lambda \mapsto (\lambda, c).$$

All balls look the same in a NLS

- The open balls of a seminormed space X are all obtained from the open unit ball, by translation and dilation:

$$B_r^X(a) = (T_a \circ D_r)(B_1^X(0)) = a + rB_1^X(0).$$

  Similarly for closed balls.

- For a surprise (?) I recommend S. Wagon, *The Banach–Tarski Paradox* (Cambridge University Press, 1987).

Definition

A subset A of a vector space V over $\mathbb{K}$ is:

- *absorbing* if $\bigcup_{t>0} tA = V$;
- *balanced* if $B_1^{\mathbb{K}}[0]A = A$;

The *Minkowski functional*, of an absorbing, balanced convex set A, is the function

$$\mu_A : V \to [0, \infty[, \quad x \mapsto \inf\{s > 0 : x \in sA\}.$$

Proposition

Let V be a vector space over $\mathbb{K}$.

(a) *Let A be the open unit ball w.r.t. a seminorm p on V. Then A is absorbing, balanced and convex, and its Minkowski functional is p.*

(b) *Conversely, let A be an absorbing, balanced convex subset of V. Then its Minkowski functional μ_A is a seminorm whose balls satisfy*

$$B_1(0) \subset A \subset B_1[0] \quad \textit{and} \quad \mu_{B_1(0)} = \mu_A = \mu_{B_1[0]}.$$

Proof.

Exercise. □

TVS-2

- Vector topologies, continuity of T_a, D_μ and $|c\rangle$; local bases
- Direct products of TVS's
- Basic facts; regularity of TVS's
- Characterisation of vector topologies
- More basic facts; a local base of closed, balance sets

Definition

A *vector topology* for a vector space V over $\mathbb{K}$ is a topology on V with respect to which summation and scalar multiplication are continuous. A *topological vector space* (TVS) is a vector space with a vector topology.

Example

Seminormed spaces are topological vector spaces; normed spaces are Hausdorff topological vector spaces.

Proposition

Let $a \in X$ and $\mu \in \mathbb{K} \setminus \{0\}$ for a TVS X over $\mathbb{K}$. Then T_a and D_μ are homeomorphisms and $|a\rangle$ is continuous. In particular,

$$\mathcal{N}(a) = \mathcal{N}(0) + a.$$

Proof.

This follows from the continuity of the maps f_a, g_μ and h_a just introduced. $\square$

Terminology/Notation

Neighbourhood bases at 0 are referred to as *local bases*; we shall write $\mathcal{N}_X$ for $\mathcal{N}(0_X)$.

Proposition

Let X be the direct product of a family $(X_\gamma)_{\gamma\in\Gamma}$, of TVS's over $\mathbb{K}$. Then the product topology on X is a vector topology.

Proof.

Let $U \in \prod_{\gamma\in\Gamma} U_\gamma$ be a basic open set in X. Then

$$s^{-1}(U) = \sigma^{-1}\Big(\prod_{\gamma\in\Gamma} s_\gamma^{-1}(U_\gamma)\Big) \quad \text{and} \quad m^{-1}(U) = g^{-1}\Big(\prod_{\gamma\in\Gamma} m_\gamma^{-1}(U_\gamma)\Big),$$

where σ is the permutation map $X \times X \to \prod_{\gamma\in\Gamma}(X_\gamma \times X_\gamma)$, and g is the map

$$\mathbb{K} \times X \to \prod_{\gamma\in\Gamma}(\mathbb{K} \times X_\gamma), \quad (\lambda, x) \mapsto \Big((\lambda, x_\gamma)\Big)_{\gamma\in\Gamma}.$$

It is not hard to see that σ and g are both continuous. The result follows. □

Corollary

For each $\delta \in \Gamma$, the coordinate projection $\mathcal{P}_\delta : X \to X_\delta$ is linear, continuous and open, and the coordinate injections $I_\delta : X_\delta \to X$ induce linear homeomorphisms $\widetilde{I}_\delta : X_\delta \to \widetilde{X}_\delta := \{x \in X : x_\gamma = 0$ unless $\gamma = \delta\}$.

Proposition

Let $A, H, G, K, J, F \subset X$ *and* $\Lambda \subset \mathbb{K}$*, for a TVS* X *over* $\mathbb{K}$*, with* $H \in \mathcal{N}_X$*,* G *open,* K *and* J *compact and* F *closed. Then:*

(a) H *is absorbing;*

(b) $K + J$ *is compact;*

(c) $A + G$ *is open;*

(d) *if* $0 \notin \Lambda$ *then* ΛG *is open; if* $0 \in \Lambda$ *then* $\Lambda G = U \cup \{0\}$*, where* U *is the open set* $(\Lambda \setminus \{0\})G$*;*

(e) *there is* $U \in \mathcal{N}_X$ *open and balanced such that* $U + U \subset H$*;*

(f) *if* $K \cap F = \emptyset$ *then there is* $U \in \mathcal{N}_X$ *such that* $(K + U) \cap (F + U) = \emptyset$*;*

(g) $F + K$ *is closed*

Corollary

Let X *be a TVS. Then* X *is regular. In particular,* X *is Hausdorff if and only if* $\{0\}$ *is closed.*

Proof.

Apply (f) with K being a singleton set. □

Proof

Let $\mathcal{N} = \mathcal{N}_X$. (a) and (b) follow from the continuity at $0_{\mathbb{K}}$ of each map $|x\rangle \in L(\mathbb{K}; X)$, continuity at $(0,0)_{X\times X}$ of summation, and compactness of $K \times J$. (c) and (d) follow from the identities

$$A + G = \bigcup_{a\in A} T_a(G) \quad \text{and} \quad \Lambda G = \bigcup_{\lambda\in\Lambda} D_\lambda(G).$$

(e) By continuity of summation, $(0,0)_{X\times X}$ has a basic neighbourhood $W_1 \times W_2 \in \mathcal{N} \times \mathcal{N}$, such that $W_1 + W_2 \subset H$. Letting $W = \mathrm{Int}(W_1 \cap W_2)$ it remains to show that W contains an open, balanced neighbourhood of 0_X. Continuity of scalar multiplication implies that $(0,0)_{\mathbb{K}\times X}$ has a basic neighbourhood $B_r[0] \times V$, for some $r > 0$ and open $V \in \mathcal{N}$, satisfying $U \subset W$ where $U = B_r^{\mathbb{K}}[0]V$. Since

$$0 \in (B_r^{\mathbb{K}}[0] \setminus \{0\})V \quad \text{and} \quad B_1^{\mathbb{K}}[0]B_r^{\mathbb{K}}[0] = B_r^{\mathbb{K}}[0]$$

it follows that U is open and balanced.

Proof continued.
(f) For $x \in K$, applying (e) twice there is $U_x \in \mathcal{N}$ open, balanced and satisfying $x + U_x + U_x + U_x \subset F^c$. By compactness there is $K_0 \subset\subset K$ such that $K \subset \bigcup_{x \in K_0}(x + U_x)$. Set $U = \bigcap_{x \in K_0} U_x$ then

$$K + U + U \subset \bigcup_{x \in K_0}(x + U_x + U_x + U_x) \subset F^c.$$

Since $U = -U$ this implies that $(K + U) \cap (F + U) = \emptyset$.
(g) Let $x \in (F + K)^c$. Then $(x - F) \cap K = \emptyset$ and $(x - F) = (T_x \circ D_{-1})(F)$ is closed so, by (f), there is $H \in \mathcal{N}$ such that $(x - F + H) \cap K = \emptyset$, so $x + H \in (F + K)^c$ and (g) follows. □

The characterisation of vector topologies (in the definition) is by what they must *do* (that is, make certain maps continuous). The following characterisation is sometimes easier to verify than the axioms themselves.

Proposition

Let τ be a topology on a vector space X. Then TFAE:

(i) *τ is a vector topology;*

(ii) *τ is translation-invariant (that is, every translation is a homeomorphism) and has a local base $\mathcal{N}$ consisting of balanced and absorbing sets satisfying*

$$\forall_{H \in \mathcal{N}} \, \exists_{H' \in \mathcal{N}} \quad H' + H' \subset H.$$

Proof.

Exercise. □

Proposition

Let $A, A', B, C, Z \subset X$, for a TVS X, with B balanced, C convex and Z a subspace of X. Then:

(a) $\overline{A} = \bigcap_{H \in \mathcal{N}_X} (A + H)$;

(b) $\overline{A} + \overline{A'} \subset \overline{A + A'}$;

(c) $A + \operatorname{Int} A' \subset \operatorname{Int}(A + A')$;

(d) $\overline{B}$, $\overline{C}$ *and* $\overline{Z}$ *are respectively balanced, convex and a subspace of* X;

(e) $\operatorname{Int} B$ *is balanced if it contains* 0;

(f) *the collection of closed balanced sets in* $\mathcal{N}_X$ *form a local base;*

(g) *every convex set in* $\mathcal{N}_X$ *contains a closed convex and balanced neighbourhood of* 0.

Proof.

(g) Let $U \in \mathcal{N}_X$ be convex. Take an open balanced set $W \in \mathcal{N}_X$ satisfying $W \subset U$ and let $V = \bigcap_{|\lambda|=1} \lambda U$. Then V is convex and balanced; moreover, since W is balanced, $V \supset W$ so $V \in \mathcal{N}_X$. By (a), (c) and the convexity of V, the set $\frac{1}{2}\overline{V}$ is closed, convex and balanced, and satisfies $\frac{1}{2}\overline{V} \subset \frac{1}{2}V + \frac{1}{2}V \subset V \subset U$. The rest is left as an exercise. □

TVS-3

- Absolute convergence characterisation of completeness
- $\dim X < \infty$: automatic continuity for $T \in L(X; Y)$ if X and Y are Hausdorff
- $\dim X < \infty$: local compactness; if normed then complete and satisfying Heine–Borel

Absolute convergence characterisation of completeness

Lemma

Let d be a translation-invariant metric on a vector space V. Then TFAE:

(i) *V is d-complete;*

(ii) *any sequence (x_n) in V satisfying $\sum_{n=1}^{\infty} d(x_{n+1}, x_n) < \infty$ converges.*

Proof.

Exercise. □

Corollary

A NLS X is a Banach space if and only if all of its absolutely convergent series converge.

Lemma

Let $T \in L(\mathbb{K}^n; X)$ for $n \in \mathbb{N}$ and a Hausdorff TVS X. Then:

(a) *T is continuous;*

(b) *if T is a linear isomorphism then it is a homeomorphism.*

Proof.

(a) Let $\varphi_1, \ldots, \varphi_n$ be the dual basis of the standard basis $e_1, \ldots, e_n$ for $\mathbb{K}^n$. Then $T = \sum_{i=1}^n |Te_i\rangle\varphi_i$ is a sum of compositions of manifestly continuous maps.

(b) Let S be the unit sphere in $\mathbb{K}^n$. Then S is compact and T is a continuous bijection so $T(S)$ is a compact subset of X which does not contain 0. Since X is Hausdorff $T(S)$ is closed and so there is an open neighbourhood U of 0 disjoint from $T(S)$ which we may suppose to be balanced. Then $x \in U \implies \|T^{-1}x\| < 1$, which implies that T^{-1} is continuous (exercise). □

Remark

The proof for NLS X is only simplified by the availability of the balanced open neighbourhoods $B_{1/n}(0)$, $n \in \mathbb{N}$.

Theorem

Let X and Y be Hausdorff TVS's with X finite-dimensional. Then:

(a) *any linear map from X into Y is continuous;*

(b) *any linear isomorphism from X to Y is a homeomorphism;*

(c) *X is locally compact.*

If X is a NLS then also:

(d) *X is complete;*

(e) *any closed and bounded subset of X is compact.*

Proof.

Straightforward application of the lemma, the completeness of $\mathbb{K}^n$, the Heine–Borel Theorem (in $\mathbb{K}^n$) and the Lipschitz/boundedness of linear morphisms between NLS's (exercise). □

Remark

If Y is a finite-dimensional subspace of a NLS X then Y is complete (by (d)) and thus closed. In fact, the same holds if X is a Hausdorff TVS (where completeness refers to Cauchy nets - see TVS-5).

Theorem

Every finite dimensional subspace F of a Hausdorff TVS X is closed.

Proof.

Let $x \in \overline{F}$. It suffices to show that $x \in F$. Take a linear isomorphism $T : F \to \mathbb{K}^d$ where $d := \dim F$. Then, with respect to the relative topology of F and the standard topology of $\mathbb{K}^d$, F and $\mathbb{K}^d$ are Hausdorff TVS's and T is a homeomorphism. Let D and B denote the open and closed unit balls of $\mathbb{K}^d$ with respect to the Euclidean norm. Then, being open in F, the set $T^{-1}(D)$ equals $F \cap U$ for some open subset U of X. Since $0 \in T^{-1}(D)$, U is a neighbourhood of 0 and so is absorbing, there is therefore $t > 0$ such that $x \in tU$. Now

$$F \cap tU = t(F \cap U) = tT^{-1}(D) \subset tT^{-1}(B)$$

and $tT^{-1}(B) = T^{-1}(tB)$ is compact in F and thus compact and so closed in (the Hausdorff space) X. Therefore, since $x \in \overline{F}$ and U is open in X, $x \in \overline{F} \cap tU \subset \overline{F \cap tU} \subset tT^{-1}(B) \subset F$ so $x \in F$, as required. □

Remarks

Later we see a different proof in which the result is deduced from the completeness of F which is proved using the automatic uniform continuity of continuous linear maps between TVS's. In fact, the Hausdorff assumption is redundant. Finite dimensional subspaces of any TVS are complete and thus

TVS-4

- Quotient TVS and NLS
- Adding closed subspaces

Proposition

Let Z be a subspace of a TVS X and let Q be the quotient map $X \to X/Z$.

(a) *The quotient topology, from the equivalence relation*

$$x \sim x' \quad \text{if } (x' - x) \in Z,$$

is a vector topology for the quotient vector space X/Z and Q is an open map. Moreover, for any function $f : X/Z \to S$, into a topological space S, f is open (respectively, continuous) if and only if $f \circ Q : X \to S$ is.

(b) *X/Z is locally convex if X is.*

(c) *X/Z is Hausdorff if Z is closed.*

(d) *Suppose that X is metrisable (respectively, normable) and Z is closed. Then X/Z is metrised (respectively, normed) by*

$$\widetilde{d}([x],[x']) := \operatorname{dist}(x' - x, Z), \quad \textit{respectively } \operatorname{dist}(x, Z)$$

for any translation-invariant metric (respectively, norm) defining dist *for X. The quotient map is Lipschitz with Lipschitz constant 1 (unless $Z = X$), and X/Z is complete if and only if X is.*

$Q(B_r^X(a)) = B_r^{X/Z}([a])$

Remark

If a TVS is pseudometrisable, then it may be pseudometrised by a translation-invariant pseudometric (as we shall see later).

Proof.

(a) Openness of Q follows from the fact that if U is open in X then its $\sim$-saturation $U_\sim$ is equal to $U + Z$ which is also open in X (see Quotient Topology, Section B). The rest is left as an exercise.

(c) If Z is closed and $[x] \neq 0_{X/Z}$ then $x \notin Z$ so, for any neighbourhood U of x disjoint from Z, $Q(U)$ is a neigbourhood of $[x]$ not containing $0_{X/Z}$. It follows that X/Z is T_1 (singleton sets are closed) and thus Hausdorff.

The converse of (c) and the rest of the proof is left as an exercise. [HINT for completeness: use the 'absolute convergence' characterisation.] □

Remark

In case (d), openness of the quotient map is made manifest by the identity

$$Q\Big(B_r^X(a)\Big) = B_r^{X/Z}([a]), \quad a \in X, \; r > 0.$$

Corollary

Let Z and F be subspaces of a TVS X. If F is finite-dimensional and Z is closed then $Z + F$ is closed.

Proof.

Since $Z + F = Q^{-1}(Q(F))$ where Q is the quotient map $X \to X/Z$, the result follows from the continuity of Q and the closedness of the finite dimensional subspace $Q(F)$ of the Hausdorff TVS X/Z. □

Example (Nonclosed sum of two closed subspaces)

Let G be the graph of a continuous map $T \in L(X;Y)$ with nonclosed range I, for Hausdorff TVS's X and Y. (For example, if $X = Y = \ell^2$ and $T : (x_n) \mapsto (\frac{1}{n}x_n)$ then $I = \{(z_n) \in \ell^2 : (nz_n) \in \ell^2\}$ which is a dense but proper subspace of ℓ^2.) Then G and $X \oplus \{0\}$ are closed subspaces of $X \oplus Y$, but

$$G + (X \oplus \{0\}) = X \oplus I,$$

which is nonclosed.

TVS-5

- Uniform spaces; bases for uniformity
- Uniform topologies
- Vicinity and neighbourhood
- Uniform continuity
- Uniform topology $\equiv$ completely regular topology
- Completeness for uniform spaces
- Completion and uniformly continuous extension

Definition

A *uniform space* (S, υ) is a set S together with a *uniformity* on S, that is a nonempty family υ of subsets of S such that, for all $U, V \in \upsilon$ and $A \subset S \times S$:

(Ui) $V \supset D$ where $D := \{(s,s) : s \in S\}$;

(Uii) $V^{-1} \in \upsilon$ where $V^{-1} := \{(s', s) : (s, s') \in V\}$;

(Uiii) $\exists_{V' \in \upsilon}\ V' \circ V' \subset V$ where

$$W_1 \circ W_2 := \Big\{(s, s'') : \exists_{s' \in S}\ (s, s') \in W_1 \text{ and } (s', s'') \in W_2\Big\};$$

(Uiv) $U \cap V \in \upsilon$;

(Uv) $A \supset V \implies A \in \upsilon$.

A *base* for a uniformity υ is a subset $\mathcal{B}$ of υ such that υ consists of all of the supersets of the basic sets. It is a straightforward exercise to prove:

Proposition

A nonempty family $\mathcal{B}$ of subsets of $S \times S$ is a base of a uniformity on S if and only if for all $V, V' \in \mathcal{B}$ there is $U_1, U_2, U_3 \in \mathcal{B}$ such that:

(Bi) $U_i \supset D (i = 1, 2, 3)$, (Bii) $U_1 \subset V^{-1}$, (Biii) $U_2 \circ U_2 \subset V$ and (Biv) $U_3 \subset V \cap V'$.

Examples

- $\{V \in \mathcal{P}(S \times S) : V \supset D\}$, the *discrete uniformity*.
- $\{S \times S\}$, the *trivial uniformity*.
- For a pseudometric space (E, d), the triangle inequality implies that

$$d^{-1}([0, r_1[) \circ d^{-1}([0, r_2[) \subset d^{-1}([0, r_1 + r_2[),$$

  and it is easily seen that $\{d^{-1}([0, r[) : r > 0\}$ is a base for a uniformity.

Proposition (Uniform topologies)

Let υ be a uniformity on a set S. Then, the following family of sets defines a topology on S:

$$\{G \subset S : \forall_{x \in G} \, \exists_{V \in \upsilon} \ \ V[x] \subset G\}, \quad \textit{where } V[x] := \{y \in S : (x, y) \in V\},$$

It is called the uniform topology of υ, and is Hausdorff if and only if $\bigcap \upsilon = D$.

Proof.

Exercise. □

Remark

As with metrisations, in general many uniformities yield the same topology.

Proposition

Let τ be the uniform topology of a uniformity υ on a set S. Then, for $x \in A \subset S$, TFAE:

(i) $x \in \operatorname{Int} A$;

(ii) $\exists_{V \in \upsilon}\ V[x] \subset A$.

In fact, the following identity holds:

$$\{V[x] : V \in \upsilon\} = \mathcal{N}(x).$$

Remark

We have used the letter V in deference to the name 'vicinity', for the elements of a uniformity, adopted by the pioneers. The last identity nicely connects vicinities of a uniformity with neighbourhoods of its uniform topology.

Definition

Let $f \in \mathcal{F}(S; S')$ where the sets S and S' have uniformities υ and υ'. Then f is *uniformly continuous* if

$$\text{for all } V' \in \upsilon' \quad \{(x,y) \in S \times S : (f(x), f(y)) \in V'\} \in \upsilon,$$

equivalently,

$$\forall_{V' \in \upsilon'} \, \exists_{V \in \upsilon} \; \{(f(x), f(y)) : (x,y) \in V\} \subset V'.$$

If f is bijective and both f and f^{-1} are uniformly continuous then (unfortunately) f is called a *uniform isomorphism*.

Proposition

Uniformly continuous maps between uniform spaces are continuous with respect to the corresponding uniform topologies.

Proof.

Exercise. □

Theorem

For a topological space S, TFAE:

(i) *S is the uniform topology of some uniformity on S;*

(ii) *S is completely regular;*

(iii) *S is homeomorphic to a topological subspace of $\prod_{\gamma\in\Gamma} E_\gamma$ for a family (E_γ) of pseudometrisable spaces.*

In this case, the homeomorphism may be chosen to be a uniform isomorphism. Moreover, if S is Hausdorff then each E_γ can be chosen to be metrisable.

Remark

Given an indexed family of uniform spaces $\left(S_\gamma, \upsilon_\gamma\right)_{\gamma\in\Gamma}$ the product $\prod_{\gamma\in\Gamma} S_\gamma$ has a natural uniform structure with respect to which a net $(x^\lambda)_{\lambda\in\Lambda}$ is Cauchy if and only if each $(x^\lambda_\gamma)_{\lambda\in\Lambda}$ is.

Completeness for uniform spaces

Definition

A net $(x_\lambda)_{\lambda \in \Lambda}$ in a uniform space (S, υ) is *Cauchy* if, for all $V \in \upsilon$ there is $\nu \in \Lambda$ such that $(x_\lambda, x_\mu) \in V$ for $\lambda, \mu \geq \nu$; a subset A of S is *complete* if all of its Cauchy nets converge (w.r.t. the topology of the uniformity) with limit in A.

Nets which converge (w.r.t. the topology of the uniformity) are Cauchy. We have consistency with the already-familiar notion of completeness: For a pseudometrisable uniform space S, TFAE:

- S is complete;
- S is sequentially complete: every Cauchy *sequence* converges.

Warning

First-countability alone is insufficient to deliver equivalence of completeness and sequential completeness.

Remark

Complete subsets of a uniform space are closed in the topology of the uniformity.
A product of uniform spaces is complete if each one is complete.

Completion and uniformly continuous extension

Definition

A *completion* of a uniform space S is a complete uniform space T together with a uniform isomorphism j from S onto a dense subset of T.

Theorem

Every uniform space S has a completion. If (T_1, j_1) and (T_2, j_2) are completions of S then there is a unique uniform isomorphism $j_{21} : T_1 \to T_2$ satisfying $j_{21} \circ j_1 = j_2$. If S is Hausdorff then so are its completions.

Remark

This is proved by exploiting the fact that S is uniformly isomorphic to a topological subspace of a product of a family of pseudometrisable uniform spaces and appealing to the completion of pseudometric spaces.

Theorem (Uniformly Continuous Extension)

Let $f_0 : S_0 \to T$ be a uniformly continuous function from a dense subset of a uniform space S into a complete Hausdorff uniform space T. Then f_0 has a unique uniformly continuous extension $f : S \to T$.

Remark

The proof consists of showing that the closure of the graph of f_0 in $S \times T$ is the graph of a uniformly continuous function with domain S.

TVS-6

- Translation-invariant uniformity for a TVS
- Continuous linear extension
- TVS completion

Each TVS has a unique translation-invariant uniformity

Definition

A uniformity on a vector space X is *translation-invariant* if it has a base $\mathcal{B}$ satisfying

$$\forall_{V \in \mathcal{B}}\ \forall_{(x,y) \in V}\ \forall_{a \in X}\ \ (x + a, y + a) \in V.$$

Theorem (Every TVS has a unique translation-invariant uniformity)

Let X be a TVS. Then there is a unique translation-invariant uniformity υ on X whose uniform topology is the vector topology of X. If $\mathcal{N}$ is a local base for X then the following family is a base for υ:

$$\Big\{ V_H := \{(x, y) : (y - x) \in H\} : H \in \mathcal{N} \Big\}$$

Corollary

TVS's are completely regular.

Corollary

Every continuous linear map T between TVS's is uniformly continuous.

Proof.

This follows since, for any neighbourhood H of 0 in the target,

$$\{(x, y) : (Tx, Ty) \in V_H\} \subset V_{T^{-1}(H)}.$$

□

Corollary

Every finite-dimensional subspace F of a Hausdorf TVS X is complete, and thus closed.

Proof.

Let (x_λ) be a Cauchy net in F. Take a linear isomorphism $T : F \to \mathbb{K}^d$, where $d := \dim F$. Then T is linear homeomorphism, in particular T is uniformly continuous. It follows that (Tx_λ) is Cauchy in $\mathbb{K}^d$ and thus convergent. Therefore, by the continuity of T^{-1}, $(x_\lambda = T^{-1}Tx_\lambda)$ converges too. The result follows. □

Theorem (Continuous Linear Extension)

Let $T_0 : X_0 \to Y$ be a continuous linear map from a dense subspace of a TVS X into a complete TVS Y. Then there is a unique continuous map $T : X \to Y$ extending T_0. Moreover, T is linear.

Proof.

Since T_0 is necessarily uniformly continuous, the first part follows from the Uniformly Continuous Extension Theorem. Uniqueness and the second part follow from the Identity Theorem; in the latter case, applied to the function

$$X \times \mathbb{K} \times X \to Y, \quad (x, \lambda, x') \mapsto T(x + \lambda x') - Tx - \lambda Tx',$$

which agrees with the zero function on the dense subset $X_0 \times \mathbb{K} \times X_0$. □

Definition

A *completion* of a TVS X is a complete TVS Z together with a continuous linear map $J : X \to Z$ with dense image whose induced map $\widetilde{J} : X \to J(X)$ is a homeomorphism.

Remark

Since linear homeomorphism are uniform isomorphisms, a TVS completion is in particular a uniform space completion.

Proposition

Every TVS X has a completion. If X is Hausdorff then it has a Hausdorff completion. If (Z_1, J_1) and (Z_2, J_2) are completions of a Hausdorff TVS then there is a unique linear homeomorphism $J_{21} : Z_1 \to Z_2$ such that $J_{21} \circ J_1 = J_2$.

TVS-7

- TVS types: locally convex, F-space, Fréchet spaces
- Local convexity and seminorms
- Pseudometrisable $\equiv$ first countable, for a TVS
- X-boundedness, for a TVS
- Kolmogorov's Normability Theorem
- TVS $X \hookrightarrow \prod_{\gamma \in \Gamma} Z_\gamma$, for pseudometrisable TVS's (Z_γ)
- Locally convex $X \hookrightarrow \prod_{\gamma \in \Gamma} Z_\gamma$, for seminormable TVS's (Z_γ)
- Metrisability/normability of direct products

Definition

Let X be a TVS. Then X is:

- *locally convex* if every neighbourhood of 0_X contains a convex neighbourhood;
- (*semi-*)*normable* if its topology is induced by a (semi-)norm;
- (*pseudo-*)*metrisable* if its topology is induced by a (pseudo-)metric;
- an *F-space* if it is completely metrisable by a translation-invariant metric;

A *Fréchet space* is a locally convex F-space.

Remarks

- X is locally convex if and only if the convex neighbourhoods of 0_X form a local base.
- Since open balls of a seminormed space are convex, and open balls centred at 0 form a local base, seminormable spaces are locally convex.
- Banach spaces are Fréchet.

Proposition

Let $\mathcal{P}$ be a family of seminorms on a vector space V. Then the topology with subbase

$$\left\{p^{-1}(B^{\mathbb{R}}_{1/n}(0)) = \{x \in V : p(x) < 1.n\} \middle| p \in \mathcal{P},\ n \in \mathbb{N}\right\}$$

is a locally convex topology τ, with respect to which each seminorm $p \in \mathcal{P}$ is continuous. Moreover, τ is Hausdorff if and only if $\mathcal{P}$ is separating:

$$\forall_{x \in V}\ \exists_{p \in \mathcal{P}}\quad p(x) \neq 0.$$

If $\mathcal{P}$ is countable then, for any enumeration (p_n) of $\mathcal{P}$,

$$d(x, y) := \sum_{n \geq 1} 2^{-n}(\alpha \circ p_n)(y - x), \quad \textit{where } \alpha(t) = t/(1 + t),$$

defines a translation-invariant metric for τ.

Warning

Surprisingly, the open d-balls centred at 0_V may *not* be convex.

Example

Let $C = C(\mathbb{K}^d)$ with the topology of uniform convergence on compacts. Then C is a Fréchet space since

$$d(f,g) := \sum_{n \geqslant 1} 2^{-n} \alpha\Big(\big\| (g-f)|_{B_n[0]} \big\|_{\sup} \Big) \quad \text{where } \alpha : t \mapsto t/(1+t)$$

defines a complete translation-invariant metric for C whose balls are convex.

C is *not* normable.

Theorem

Let X be a TVS. Then TFAE:

(i) *X is pseudometrisable;*

(ii) *X is first countable.*

In this case, X is pseudometrised by a translation-invariant pseudometric whose balls centred at 0_X are balanced – and also convex if X is locally convex.

Lemma/Definition

For a subset A of a TVS X, TFAE:

(i) $\forall_{H \in \mathcal{N}_X}\ \exists_{s \in \mathbb{R}_+}\ \forall_{t > s}\ \ tH \supset A$;

(ii) $\forall_{H \in \mathcal{N}_X}\ \exists_{n \in \mathbb{N}}\ \ nH \supset A$;

(Exercise.) Such sets are called X-bounded.

An open neighborhood H of 0 is X-bounded if and only if $(n^{-1}H)_{n \in \mathbb{N}}$ is a local base for X.

Examples

- Compact subsets of a TVS X are X-bounded.
- Balls of a seminormed space X are X-bounded.

Warning

- X itself cannot be X-bounded unless X is trivial $(= \{0\})$.
- If X is metrised by a (translation-invariant) metric d, then it is metrised by the bounded (translation-invariant) metric $d' := \alpha \circ d$, where $\alpha(t) = t/(1+t)$, so X is d'-bounded (but not X-bounded).

Theorem (Kolmogorov)

Let X be a TVS. Then TFAE:

(i) *X is seminormable;*

(ii) *0_X has an X-bounded convex neighbourhood.*

Proof.

Let $H \in \mathcal{N}_X$ be X-bounded and convex. Then there is $B \in \mathcal{N}_X$ balanced, convex and contained in H. The Minkowski functional of B is a seminorm p satisfying

$$B_1(0) \subset B \subset B_1[0].$$

Thus $(\frac{1}{n}B)_{n\in\mathbb{N}}$ is a local base for the seminorm topology, as well as for the original vector topology (by X-boundedness). Therefore the topologies coincide and X is seminormed by p. The converse has already been noted. □

Remark

X is normable if it is seminormable and Hausdorff.

Theorem

Any TVS is linearly homeomorphic to a TVS-subspace of a direct product of pseudometrisable spaces.

Remark

This brings together earlier results on uniform topology, complete regularity and uniformisability of TVS's.

Theorem

Let X be a TVS. Then TFAE:

(i) *X is locally convex;*

(ii) *X is linearly homeomorphic to a TVS subspace of a direct product of seminormed spaces.*

If X is Hausdorff then the spaces may be chosen to be normed.

Theorem

Let X be the direct product of a family $(X_\gamma)_{\gamma\in\Gamma}$ of TVS's.

(a) If each X_γ has nontrivial topology then TFAE:

(i) *X is pseudometrisable itself;*

(ii) *Γ is countable and each X_γ is pseudometrisable.*

(b) TFAE:

(i) *X is seminormable itself;*

(ii) *there is $\Gamma_0 \subset\subset \Gamma$ such that X_γ is seminormable for $\gamma \in \Gamma_0$ and has trivial topology for $\gamma \in \Gamma \setminus \Gamma_0$.*