Lecture 6: Left invariant vector fields and one-parameter subgroups: Examples and Consequences

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The space of left (right) invariant vector fields is naturally isomorphic (as a vector space) to the tangent space $T_eG$ at the identity.

Left (right) invariant vector fields are complete.

Integral curves of left (right) invariant vector fields through the identity $e \in G$ are exactly one-parameter subgroups.

The flow on $G$ generated by a left invariant vector field $\xi$ can be written in the form $\Phi^t_\xi(a) = a \cdot \exp(t \xi_0)$, where $\xi_0 = \xi(e)$.

Any left invariant vector field commute with any right invariant vector field.

The space of left invariant vector fields is closed under the Lie bracket.
Important remark: Let $G \subset GL(n, \mathbb{R})$ be a matrix Lie group. Then a tangent vector to $G$ at $X_0 \in G$ can (and will) be considered as a certain $n \times n$ matrix. We use our usual idea: tangent vectors to $G$ are tangent vectors to curves lying in $G$. A curve passing through $X_0$ is a family of matrices $X(t) \in G$ (smoothly depending on $t$), $X(0) = X_0$, so its derivative w.r.t. $t$ is again an $n \times n$ matrix:

$$X(t) = \begin{pmatrix} x_{11}(t) & \cdots & x_{1n}(t) \\ \vdots & \ddots & \vdots \\ x_{n1}(t) & \cdots & x_{nn}(t) \end{pmatrix} \quad \rightarrow \quad \frac{dX}{dt}(0) = \begin{pmatrix} x'_{11}(0) & \cdots & x'_{1n}(0) \\ \vdots & \ddots & \vdots \\ x'_{n1}(0) & \cdots & x'_{nn}(0) \end{pmatrix} \in T_{X_0} G$$

The tangent space $T_X G$ is a certain subspace in $M_{n,n}$ (space of all matrices) which depends on $X$. However, if $G = GL(n, \mathbb{R})$, then the tangent space $T_X GL(n, \mathbb{R})$ coincides with $M_{n,n}$ at each point $X$.

Proposition

Let $A$ be an arbitrary $n \times n$-matrix viewed as a tangent vector to $GL(n, \mathbb{R})$ at $E = Id$. Then the corresponding left invariant vector field on $GL(n, \mathbb{R})$ is $\xi(X) = XA$.

Proof. By construction, $\xi(X) = dL_X(A) = XA$ (left multiplication by $X$ is a linear map, so it coincides with its own differential).
Proposition
Let $A$ be an arbitrary $n \times n$-matrix viewed as a tangent vector to $GL(n, \mathbb{R})$ at $E = Id$. Then the corresponding one-parameter subgroup is given as

$$\exp(tA) = e^{tA} = E + tA + \frac{t^2 A^2}{2!} + \frac{t^3 A^3}{3!} + \ldots$$

Proof. First of all, it is an easy exercise to check that this series converges for any $A$ and $t$ absolutely (and uniformly on any interval $t \in (-T, T)$). Moreover, the resulting matrix function is smooth w.r.t. $t$ and

$$\frac{d}{dt} e^{tA} = Ae^{tA} = e^{tA} A. \tag{1}$$

Besides,

$$e^{(t+s)A} = e^{tA} \cdot e^{sA}. \tag{2}$$

Reminder: $e^A e^B \neq e^{A+B}$ if $A \neq B$.

Now there are two ways to complete the proof.

- (1) means that $e^{tA}$ is an integral curve of the left invariant vector field $\xi(X) =XA$ through the identity. Indeed, $\frac{d}{dt} e^{tA} = e^{tA} A = \xi(e^{tA})$ as needed.

- (2) means that $e^{tA}$ is a one-parameter subgroup (with the initial tangent vector $A$).
The case of matrix groups: arbitrary \( G \subset GL(n, \mathbb{R}) \)

Natural question: What are analogs of these two statements in the case of an arbitrary matrix Lie group \( G \subset GL(n, \mathbb{R}) \)?

Answer: These statements holds for any \( G \subset GL(n, \mathbb{R}) \) without any change.

This immediately follows from the uniqueness condition: for any tangent vector \( \xi_0 \in T_e G \), there is a unique one-parameter subgroup \( f : \mathbb{R} \to G \) such that \( \frac{df}{dt}(0) = \xi_0 \) and there is a unique left invariant vector field \( \xi \) such that \( \xi(e) = \xi_0 \).

Corollary

Let \( g = T_E G \subset M_{n,n} \) be the tangent space of a matrix group \( G \subset GL(n, \mathbb{R}) \) at the identity. Then the tangent space at any other point \( X \in G \) is:

\[
T_X G = X \cdot g = g \cdot X
\]

Corollary

In the above notation, consider a system of linear ODE

\[
\frac{dx}{dt} = A(t)x, \quad x \in \mathbb{R}^n.
\]

Let \( A(t) \in g \) for any \( t \in \mathbb{R} \). Then the fundamental solution \( X(t) \) belongs to \( G \) for any \( t \in \mathbb{R} \) (recall that by the fundamental solution we mean \( X'(t) = A(t)X(t) \) and \( X(0) = E \)).
Left-invariant vector fields for a matrix Lie group

\begin{center}
\begin{tikzpicture}
  \node at (0,0) [circle,fill,inner sep=1.5pt] {};
  \draw[->] (0,0) -- (2,2) node [above] {$A = \xi(E)$};
  \draw[->] (0,0) -- (2,-2) node [below] {$\exp tA$};
  \draw[->] (2,2) -- (4,4) node [above] {$\xi(x) = xA$};
  \draw[->] (2,-2) -- (4,-4) node [below] {$\exp tA$};

  \node at (7,0) [circle,fill,inner sep=1.5pt] {};
  \draw[->] (7,0) -- (9,2) node [above] {$\xi_0 = dL_{x_0}(\xi_0)$};
  \draw[->] (7,0) -- (9,-2) node [below] {$\xi(x) = dL_x(A) = xA$};

  \node at (12,0) [circle,fill,inner sep=1.5pt] {};
  \draw[->] (12,0) -- (14,2) node [above] {$X \in G$};
  \draw[->] (12,0) -- (14,-2) node [below] {$A \in T_E G$};

  \node at (0,-3) {$E = \text{Id}$};
  \node at (7,-3) {$x \in G$};
  \node at (12,-3) {$A \in T_E G$};

  \node at (4,-4) {$\mathbf{GL}(n, \mathbb{R})$};
\end{tikzpicture}
\end{center}

**Figure**: Left-invariant vector fields for a matrix Lie group
Tangent spaces of a matrix group

Figure: Tangent spaces of a matrix group
**Definition**

The exponential map \( \exp : T_e G \rightarrow G \) is defined by

\[
\exp(\xi_0) = \exp(t\xi_0)|_{t=1},
\]

where \( \xi_0 \in T_e G \) and \( \exp(t\xi_0) \), as before, denotes the one-parameter subgroup in \( G \) with the initial vector \( \xi_0 \).

**Remark**

Our notation \( \exp(t\xi_0) \) can now be understood in two different ways: as the image under "exp" of the tangent vector \( t\xi_0 \) or as the point on the one-parameter subgroup \( \exp(t\xi_0) \) with parameter \( t \). In fact, these points coincide (check it!), so the notation causes no confusion.

**Properties of the exponential map:**

- \( \exp \) is smooth and globally defined on \( T_e G \) as a whole;
- the differential of \( \exp \) at zero is the identity operator:
  \[
d \exp : T_e G \rightarrow T_e G, \quad d \exp(\xi_0) = \xi_0;
\]
- \( \exp \) is a local diffeomorphism at a neighborhood of zero.
Why $d \exp|_0 = \text{id}$?

$$\exp : T_e G \to G, \quad \exp(0) = e,$$

where $0 \in T_e G$ is the zero vector.

$$d \exp|_0 : T_0(T_e G) \to T_e G$$

Notice that $T_e G$ is a vector space and therefore the tangent space to it can be identified with $T_e G$ itself, that is $T_0(T_e G) = T_e G$.

Recall that for $f : M \to N$, its differential $df|_P : T_P M \to T_{f(P)} N$ is defined by

$$df|_P \left( \frac{d\gamma}{dt} (0) \right) = \frac{d}{dt} \bigg|_{t=0} f(\gamma(t)), \quad \text{where } \gamma(0) = P.$$

In our case, for $\xi \in T_e G$, we set $\gamma(t) = t\xi \in T_e G$ and then:

$$d \exp|_0(\xi) = d \exp \left( \frac{d\gamma}{dt} (0) \right) = \frac{d}{dt} \bigg|_{t=0} \exp(t\xi) = \xi,$$

that is $d \exp|_0 = \text{id}$. 
Parallelizability

Left (or right) translation allows us to identify the tangent space \( T_xG \) at an arbitrary point \( x \in G \) with \( T_eG \). This identification is natural and implies the following topological property of Lie groups.

**Theorem**

*Any Lie group \( G \) is parallelizable, i.e., its tangent bungle \( TG \) is trivial:*

\[
TG \cong G \times \mathbb{R}^n, \quad (n = \dim G).
\]

**Proof.** In general, the triviality of the (tangent) bundle \( TM \) means that there is a smooth map

\[
\phi : M \times \mathbb{R}^n \to TM
\]

which is linear on each fiber, i.e., \( \phi(x, \xi) = (x, A_x(\xi)) \), where \( A_x : \mathbb{R}^n \to T_xM \) is a linear isomorphism (that identifies \( T_xM \) with a fixed vector space \( \mathbb{R}^n \)). In our case, such a map is given by

\[
\phi : G \times \mathbb{R}^n \to TG, \quad \phi(x, \xi_0) = (x, dL_x(\xi_0)), \quad \xi_0 \in T_eG = \mathbb{R}^n.
\]

**Corollary**

- Any Lie group \( G \) is orientable.
- Among closed 2-dim surfaces, only the torus \( T^2 \) may carry (and indeed, carries) the structure of a Lie group.
Lie bracket of left-invariant vector fields for matrix Lie groups

We have not explained any relationship between Lie groups and Lie algebras yet. Here is the first example which demonstrate this relationship explicitly. (This issue will be discussed in detail at the next lecture.)

**Proposition**

Let $G \in GL(n, \mathbb{R})$ be a matrix Lie group. Let $A, B \in \mathfrak{g} = T_E G$ and $\xi(X) = XA$, $\eta(X) = XB$ be the corresponding left invariant vector fields on $G$. Then the Lie bracket of $\xi$ and $\eta$ is the left invariant vector field of the form $X(AB - BA)$.

**Corollary**

The tangent space at the identity $\mathfrak{g} = T_E G$ of any matrix Lie group is closed under the matrix commutator:

$$A, B \mapsto [A, B] = AB - BA.$$  

**Proof.** Notice that it suffices to prove the formula $[\xi, \eta] = X(AB - BA)$ for $GL(n, \mathbb{R})$ only, then it will hold for any Lie subgroup $G \subset GL(n, \mathbb{R})$ automatically. In local coordinates, the proof is straightforward (as local coordinates we just take matrix coefficients $x_{ij}$):

$$[\xi, \eta]_{ij} = \sum_{k,l} \left( \xi_{kl} \frac{\partial \eta_{ij}}{\partial x_{kl}} - \eta_{kl} \frac{\partial \xi_{ij}}{\partial x_{kl}} \right).$$
We have $\xi_{kl} = \sum_{\alpha} x_{k\alpha} a_{\alpha l}$ and, similarly, $\eta_{kl} = \sum_{\alpha} x_{k\alpha} b_{\alpha l}$.

Here by $\xi_{ij}, x_{ij}, a_{ij}$, etc. we denote the matrix coefficients of $\xi, X, A$, etc.

Hence

$$\left[ \xi, \eta \right]_{ij} = \sum_{k,l} \left( \sum_{\alpha} x_{k\alpha} a_{\alpha l} \frac{\partial}{\partial x_{kl}} \left( \sum_{\beta} x_{i\beta} b_{\beta j} \right) - \cdots \right) =$$

(It is seen that the partial derivative does not vanish in the only case $(kl) = (i \beta)$ and in this case it equals simply to $b_{\beta j}$. Hence, replacing $k$ by $i$ and $l$ by $\beta$, we have:)

$$= \sum_{\alpha, \beta} \left( x_{i\alpha} a_{\alpha \beta} b_{\beta j} - x_{i\alpha} b_{\alpha \beta} a_{\beta j} \right) =$$

$$= \sum_{\alpha, \beta} x_{i\alpha} \left( a_{\alpha \beta} b_{\beta j} - b_{\alpha \beta} a_{\beta j} \right)$$

But this formula means exactly that $\left[ \xi, \eta \right](X) = X(AB - BA)$, as stated.